

UNIVERSITY OF ARIZONA



39001006918034







Digitized by the Internet Archive  
in 2022 with funding from  
Kahle/Austin Foundation

# ELLIPTIC FUNCTIONS.

AN ELEMENTARY TEXT-BOOK FOR  
STUDENTS OF MATHEMATICS.

BY

ARTHUR L. BAKER, C.E., PH.D.,

HEAD OF DEPARTMENT OF MATHEMATICS, MANUAL TRAINING HIGH SCHOOL, BROOKLYN, N. Y.;  
FORMERLY PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF ROCHESTER, N. Y.

---

$$\sin \operatorname{am} u = \frac{1}{\sqrt{k}} \cdot \frac{H(u)}{\Theta(u)}.$$

---

FIRST EDITION.

FIRST THOUSAND.

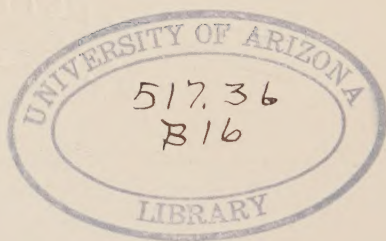
NEW YORK:

JOHN WILEY & SONS.

LONDON: CHAPMAN & HALL, LIMITED.

1906

Copyright, 1890,  
BY  
ARTHUR L. BAKER.



ROBERT DRUMMOND, PRINTER, NEW YORK





## PREFACE.

---

IN the works of Abel, Euler, Jacobi, Legendre, and others, the student of Mathematics has a most abundant supply of material for the study of the subject of Elliptic Functions.

These works, however, are not accessible to the general student, and, in addition to being very technical in their treatment of the subject, are moreover in a foreign language.

It is in the hope of smoothing the road to this interesting and increasingly important branch of Mathematics, and of putting within reach of the English student a tolerably complete outline of the subject, clothed in simple mathematical language and methods, that the present work has been compiled.

New or original methods of treatment are not to be looked for. The most that can be expected will be the simplifying of methods and the reduction of them to such as will be intelligible to the average student of Higher Mathematics.

I have endeavored throughout to use only such methods as are familiar to the ordinary student of Calculus, avoiding those methods of discussion dependent upon the properties of double periodicity, and also those depending upon Functions of Complex Variables. For the same reason I have not carried the discussion of the  $\Theta$  and  $H$  functions further.

Among the minor helps to simplicity is the use of zero subscripts to indicate decreasing series in the Landen Transformation, and of numerical subscripts to indicate increasing series. I have adopted the notation of Gudermann, as being more simple than that of Jacobi.

I have made free use of the following works: JACOBI'S *Fundamenta Nova Theoriæ Func. Ellip.*; HOUEL'S *Calcul Infinitésimal*; LEGENDRE'S *Traité des Fonctions Elliptiques*; DUREGE'S *Theorie der Elliptischen Functionen*; HERMITE'S *Théorie des Fonctions Elliptiques*; VERHULST'S *Théorie des Fonctions Elliptiques*; BERTRAND'S *Calcul Intégral*; LAURENT'S *Théorie des Fonctions Elliptiques*; CAYLEY'S *Elliptic Functions*; BYERLY'S *Integral Calculus*; SCHLOMILCH'S *Die Höheren Analysis*; BRIOT ET BOUQUET'S *Fonctions Elliptiques*.

I have refrained from any reference to the Gudermann or Weierstrass functions as not within the scope of this work, though the Gudermannians might have been interesting examples of verification formulæ. The arithmetico-geometrical mean, the march of the functions, and other interesting investigations have been left out for want of room.

# CONTENTS.

---

|                                                       |   |
|-------------------------------------------------------|---|
| INTRODUCTORY CHAPTER, . . . . .                       | o |
| CHAP. I. ELLIPTIC INTEGRALS, . . . . .                | o |
| II. ELLIPTIC FUNCTIONS, . . . . .                     | o |
| III. PERIODICITY OF THE FUNCTIONS, . . . . .          | o |
| IV. LANDEN'S TRANSFORMATION, . . . . .                | o |
| V. COMPLETE FUNCTIONS, . . . . .                      | o |
| VI. EVALUATION FOR $\phi$ , . . . . .                 | o |
| VII. FACTORIZATION OF ELLIPTIC FUNCTIONS, . . . . .   | o |
| VIII. THE $\Theta$ FUNCTION, . . . . .                | o |
| IX. THE $\Theta$ AND $H$ FUNCTIONS, . . . . .         | o |
| X. ELLIPTIC INTEGRALS OF THE SECOND ORDER, . . . . .  | o |
| XI. ELLIPTIC INTEGRALS OF THE THIRD ORDER, . . . . .  | o |
| XII. NUMERICAL CALCULATIONS, $q$ , . . . . .          | o |
| XIII. NUMERICAL CALCULATIONS, $K$ , . . . . .         | o |
| XIV. NUMERICAL CALCULATIONS, $u$ , . . . . .          | o |
| XV. NUMERICAL CALCULATIONS, $\phi$ , . . . . .        | o |
| XVI. NUMERICAL CALCULATIONS, $E(k, \phi)$ , . . . . . | o |
| XVII. APPLICATIONS, . . . . .                         | o |



# ELLIPTIC FUNCTIONS.

---

## INTRODUCTORY CHAPTER.\*

THE first step taken in the theory of Elliptic Functions was the determination of a relation between the amplitudes of three functions of either order, such that there should exist an algebraic relation between the three functions themselves of which these were the amplitudes. It is one of the most remarkable discoveries which science owes to Euler. In 1761 he gave to the world the complete integration of an equation of two terms, each an elliptic function of the first or second order, not separately integrable.

This integration introduced an arbitrary constant in the form of a third function, related to the first two by a given equation between the amplitudes of the three.

In 1775 Landen, an English mathematician, published his celebrated theorem showing that any arc of a hyperbola may be measured by two arcs of an ellipse, an important element of the theory of Elliptic Functions, but *then* an isolated result. The great problem of comparison of Elliptic Functions of different moduli remained unsolved, though Euler, in a measure, exhausted the comparison of functions of the same modulus. It was completed in 1784 by Lagrange, and for the computation

---

\* Condensed from an article by Rev. Henry Moseley, M.A., F.R.S., Prof. of Nat. Phil. and Ast., King's College, London.

of numerical results leaves little to be desired. The value of a function may be determined by it, in terms of increasing or diminishing moduli, until at length it depends upon a function having a modulus of zero, or unity.

For all practical purposes this was sufficient. The enormous task of calculating tables was undertaken by Legendre. His labors did not end here, however. There is none of the discoveries of his predecessors which has not received some perfection at his hands; and it was he who first supplied to the whole that connection and arrangement which have made it an independent science.

The theory of Elliptic Integrals remained at a standstill from 1786, the year when Legendre took it up, until the year 1827, when the second volume of his *Traité des Fonctions Elliptiques* appeared. Scarcely so, however, when there appeared the researches of Jacobi, a Professor of Mathematics in Königsberg, in the 123d number of the *Journal of Schumacher*, and those of Abel, Professor of Mathematics at Christiania, in the 3d number of *Crelle's Journal* for 1827.

These publications put the theory of Elliptic Functions upon an entirely new basis. The researches of Jacobi have for their principal object the development of that general relation of functions of the first order having different moduli, of which the scales of Lefrange and Legendre are particular cases.

It was to Abel that the idea first occurred of treating the Elliptic Integral as a function of its amplitude. Proceeding from this new point of view, he embraced in his speculations all the principal results of Jacobi. Having undertaken to develop the principle upon which rests the fundamental proposition of Euler establishing an algebraic relation between three functions which have the same moduli, dependent upon a certain relation of their amplitudes, he has extended it from three to an indefinite number of functions; and from Elliptic Functions to an infinite number of other functions embraced under an indefinite number of classes, of which that of Elliptic Func-

tions is but one ; and each class having a division analogous to that of Elliptic Functions into three orders having common properties.

The discovery of Abel is of infinite moment as presenting the first step of approach towards a more complete theory of the infinite class of ultra elliptic functions, destined probably ere long to constitute one of the most important of the branches of transcendental analysis, and to include among the integrals of which it effects the solution some of those which at present arrest the researches of the philosopher in the very elements of physics.

## CHAPTER I.

### ELLIPTIC INTEGRALS.

THE integration of irrational expressions of the form

$$Xdx \sqrt{A + Bx + Cx^2},$$

or

$$\frac{Xdx}{\sqrt{A + Bx + Cx^2}},$$

$X$  being a rational function of  $x$ , is fully illustrated in most elementary works on Integral Calculus, and shown to depend upon the transcendentials known as logarithms and circular functions, which can be calculated by the proper logarithmic and trigonometric tables.

When, however, we undertake to integrate irrational expressions containing higher powers of  $x$  than the square, we meet with insurmountable difficulties. This arises from the fact that the integral sought depends upon a new set of transcendentials, to which has been given the name of *elliptic functions*, and whose characteristics we will learn hereafter.

The name of Elliptic Integrals has been given to the simple integral forms to which can be reduced all integrals of the form

$$(I) \quad V = \int F(X, R)dx,$$

where  $F(X, R)$  designates a rational function of  $x$  and  $R$ , and  $R$  represents a radical of the form

$$R = \sqrt{Ax^4 + Bx^3 + Cx^2 + Dx + E},$$

where  $A, B, C, D, E$  indicate constant coefficients.

We will show presently that all cases of Eq. (I) can be reduced to the three typical forms

$$(2) \quad \begin{aligned} & \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \\ & \int_0^x \frac{x^2 dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \\ & \int_0^x \frac{dx}{(x^2+a)\sqrt{(1-x^2)(1-k^2x^2)}}, \end{aligned}$$

which are called elliptic integrals of the first, second, and third order.

Why they are called *Elliptic* Integrals we will learn further on. The transcendental functions which depend upon these integrals, and which will be discussed in Chapter IV, are called *Elliptic Functions*.

The most general form of Eq. (I) is

$$(3) \quad V = \int \frac{A + BR}{C + DR} dx;$$

where  $A, B, C$ , and  $D$  stand for rational integral functions of  $x$ .

$\frac{A + BR}{C + DR}$  can be written

$$\begin{aligned} \frac{A + BR}{C + DR} &= \frac{AC - BDR^2}{C^2 - D^2R^2} - \frac{(AD - CB)R^2}{C^2 - D^2R^2} \cdot \frac{1}{R} \\ &= N - \frac{P}{R}; \end{aligned}$$

$N$  and  $P$  being rational integral functions of  $x$ . Whence Eq. (3) becomes

$$(4) \quad V = \int N dx - \int \frac{P dx}{R}.$$

Eq. (4) shows that the most general form of  $V$  can be made to depend upon the expressions

$$(5) \quad V' = \int \frac{P dx}{R},$$

and

$$\int N dx.$$

This last form is rational, and needs no discussion here.

We can write

$$\begin{aligned} P &= \frac{G_0 + G_1 x + G_2 x^2 + \dots}{H_0 + H_1 x + H_2 x^2 + \dots} \\ &= \frac{G_0 + G_2 x^2 + G_4 x^4 + \dots + (G_1 + G_3 x^2 + \dots)x}{H_0 + H_2 x^2 + H_4 x^4 + \dots + (H_1 + H_3 x^2 + \dots)x}. \end{aligned}$$

Multiplying both numerator and denominator by

$$H_0 + H_2 x^2 + H_4 x^4 + \dots - (H_1 + H_3 x^2 + H_5 x^4 + \dots)x,$$

we have a new numerator which contains only powers of  $x^2$ . The result takes the following form:

$$\begin{aligned} P &= \frac{M_0 + M_2 x^2 + M_4 x^4 + \dots + (M_1 + M_3 x^2 + M_5 x^4 + \dots)x}{N_0 + N_2 x^2 + N_4 x^4 + N_6 x^6 + \dots} \\ &= \Phi(x^2) + \Psi(x^2) \cdot x. \end{aligned}$$

Equation (5) thus becomes

$$(6) \quad V' = \int \frac{\Phi(x^2) dx}{R} + \int \frac{\Psi(x^2) \cdot x \cdot dx}{R}.$$

We shall see presently that  $R$  can always be assumed to be of the form

$$\sqrt{(1-x^2)(1-k^2x^2)}.$$

Therefore, putting  $x^2 = z$ , the second integral in Eq. (6) takes the form

$$\frac{1}{2} \int \frac{\Psi(z) \cdot dz}{\sqrt{(1-z)(1-k^2z)}},$$

which can be integrated by the well-known methods of Integral Calculus, resulting in logarithmic and circular transcendents.

There remains, therefore, only the form

$$\int \frac{\Phi(x^2)dx}{R}$$

to be determined.

We will now show that  $R$  can always be assumed to be in the form

$$\sqrt{(1-x^2)(1-k^2x^2)}.$$

We have

$$\begin{aligned} R &= \sqrt{Ax^4 + Bx^3 + Cx^2 + Dx + E} \\ &= \sqrt{G(x-a)(x-b)(x-c)(x-d)}, \end{aligned}$$

$a, b, c$ , and  $d$  being the roots of the polynomial of the fourth degree, and  $G$  any number, real or imaginary, depending upon the coefficients in the given polynomial.

Substituting in equation (1)

$$x = \frac{p + qy}{1 + y},$$

we have

$$(7) \quad V = \int \phi(y, \rho) dy,$$

$\rho$  designating the radical

$$\rho = \sqrt{G[p-a+(q-a)y][p-b+(q-b)y][p-c+(q-c)y] \dots}$$

In order that the odd powers of  $y$  under the radical may disappear we must have their coefficients equal to zero; i.e.,

$$\begin{aligned}(p-a)(q-b) + (p-b)(q-a) &= 0, \\ (p-c)(q-d) + (p-d)(q-c) &= 0;\end{aligned}$$

whence

$$\begin{aligned}2pq - (p+q)(a+b) + 2ab &= 0, \\ 2pq - (p+q)(c+d) + 2cd &= 0,\end{aligned}$$

and

$$(8) \quad \begin{cases} pq = \frac{ab(c+d) - cd(a+b)}{a+b-(c+d)}, \\ p+q = \frac{2ab - 2cd}{a+b-(c+d)}. \end{cases}$$

Equation (8) shows that  $p$  and  $q$  are real quantities, whether the roots  $a, b, c$ , and  $d$  are real or imaginary;  $a, b$ , and  $c, d$  being the conjugate pairs.

Hence equation (1) can always be reduced to the form of equation (7), which contains only the second and fourth powers of the variable.

This transformation seems to fail when  $a+b-(c+d)=0$ ; but in that case we have

$$R = \sqrt{G[x^2 - (a+b)x + ab][x^2 - (a+b)x + cd]},$$

and substituting

$$x = y - \frac{a+b}{2}$$

will cause the odd powers of  $y$  to disappear as before.

If the radical should have the form

$$\sqrt{G(x-a)(x-b)(x-c)},$$

placing  $x = y^2 + a$ , we get

$$V = \int \phi(y, \rho) dy,$$

$$\rho = \sqrt{G(y^2 + a - b)(y^2 + a - c)},$$

$\phi$  designating a rational function of  $y$  and  $\rho$ .

Thus all integrals of the form contained in equation (1), in which  $R$  stands for a quadratic surd of the third or fourth degree, can be reduced to the form

$$(9) \quad V = \int \phi(x, R) dx,$$

$R$  designating a radical of the form

$$\sqrt{G(1 + mx^2)(1 + nx^2)},$$

$m$  and  $n$  designating constants.

It is evident that if we put

$$x' = x \sqrt{-m}, \quad k^2 = -\frac{n}{m},$$

we can reduce the radical to the form

$$\sqrt{(1 - x'^2)(1 - k^2 x'^2)}.$$

We shall see later on that the quantity  $k^2$ , to which has been given the name *modulus*, can always be considered real and less than unity.

Combining these results with equation (6), we see that the integration of equation (1) depends finally upon the integration of the expression

$$(10) \quad V'' = \int \frac{\phi(x'^2) dx'}{\sqrt{(1 - x'^2)(1 - k^2 x'^2)}} = \int \frac{\phi(x'^2) dx'}{R}.$$

The most general form of  $\phi(x^2)$  is

$$\begin{aligned}\phi(x^2) &= \frac{M_0 + M_2x^2 + M_4x^4 + \dots}{N_0 + N_2x^2 + N_4x^4 + \dots} \\ &= P_0 + P_2x^2 + P_4x^4 + P_6x^6 + \dots \\ &\quad + \Sigma \frac{L}{(x^2 + a)^n}.\end{aligned}$$

Hence

$$(11) \quad V'' = \Sigma P \int \frac{x^{2m} dx}{R} + \Sigma L \int \frac{dx}{(x^2 + a)^n R}.$$

But  $\int \frac{x^{2m} dx}{R}$  depends upon  $\int \frac{dx}{R}$  and  $\int \frac{x^2 dx}{R}$ , which can be shown as follows:

Differentiating  $Rx^{2m-3}$ , we have

$$\begin{aligned}d[x^{2m-3}R] &= d[x^{2m-3} \sqrt{\alpha + \beta x^2 + \gamma x^4}] \\ &= (2m-3)x^{2m-4}dx \sqrt{\alpha + \beta x^2 + \gamma x^4} \\ &\quad + \frac{x^{2m-3}(\beta x + 2\gamma x^3)dx}{\sqrt{\alpha + \beta x^2 + \gamma x^4}}.\end{aligned}$$

Integrating and collecting, we get

$$\begin{aligned}Rx^{2m-3} &= (2m-3)\alpha \int \frac{x^{2m-4}dx}{R} + (2m-2)\beta \int \frac{x^{2m-2}dx}{R} \\ &\quad + (2m-1)\gamma \int \frac{x^{2m}dx}{R} \\ (12) \quad &= \alpha' \int \frac{x^{2m-4}dx}{R} + \beta' \int \frac{x^{2m-2}dx}{R} + \gamma' \int \frac{x^{2m}dx}{R}.\end{aligned}$$

Whence we get, by taking  $m = 2$ ,

$$(13) \quad Rx = \alpha \int \frac{dx}{R} + \beta \int \frac{x^2 dx}{R} + \gamma \int \frac{x^4 dx}{R},$$

which shows that the general expression  $\int \frac{x^{2m} dx}{R}$  can be found by successive calculations, when we are able to integrate the expressions

$$\int \frac{dx}{R} \quad \text{and} \quad \int \frac{x^2 dx}{R},$$

the first and second of equation (2).

We will now consider the second class of terms in eq. (11),

$$\text{viz., } \frac{Ldx}{(x^2 + a)^n R}.$$

This second term is as follows :

$$(14) \quad \Sigma \int \frac{L}{(x^2 + a)^n R} = \int \frac{Adx}{(x^2 + a)^n R} + \int \frac{Bdx}{(x^2 + a)^{n-1} R} \\ + \int \frac{Cdx}{(x^2 + a)^{n-2} R} + \dots$$

Each of these terms can be shown to depend ultimately upon terms of the form

$$\frac{x^2 dx}{R}, \quad \frac{dx}{R}, \quad \text{and} \quad \frac{dx}{(x^2 + a)R}.$$

The two former will be recognized as the two ultimate forms already discussed, the first and second of equation (2). The third is the third one of equation (2).

This dependence of equation (14) can be shown as follows :

We have

$$\begin{aligned} d\left[\frac{xR}{(x^2+a)^{n-1}}\right] &= \frac{(x^2+a)^{n-1}(xdR+Rdx)-2x^2R(n+1)(x^2+a)^{n-2}dx}{(x^2+a)^{2n-2}} \\ &= \frac{(x^2+a)(xdR+Rdx)-2x^2R(n-1)dx}{(x^2+a)^n}. \end{aligned}$$

Substituting the value of

$$R = \sqrt{\alpha + \beta x^2 + \gamma x^4} \quad \text{and} \quad dR = (\beta x + 2\gamma x^3) \frac{dx}{R},$$

we get

$$\begin{aligned} d\left[\frac{xR}{(x^2+a)^{n-1}}\right] &= \\ \frac{(x^2+a)(\beta x^2+2\gamma x^4+\alpha+\beta x^2+\gamma x^4)-2x^2(n-1)(\alpha+\beta x^2+\gamma x^4)}{(x^2+a)^n} \cdot \frac{dx}{R} \\ &= \frac{-2(n-1)\gamma \left| \begin{array}{c} 3\gamma \\ x^6 \end{array} \right| \begin{array}{c} +2\beta \\ x^4 \end{array} \left| \begin{array}{c} +2a\beta \\ x^2 + a\alpha \end{array} \right|}{(x^2+a)^n} \cdot \frac{dx}{R} \\ &= \frac{-(2n-5)\gamma x^6 - (2n-4)\beta \left| \begin{array}{c} x^4 \\ +3a\gamma \end{array} \right| - (2n-3)\alpha \left| \begin{array}{c} x^2 + a\alpha \\ -2a\beta \end{array} \right|}{(x^2+a)^n} \cdot \frac{dx}{R}; \end{aligned}$$

or, by substituting in the numerator  $x^2 = z - a$ ,

$$\begin{aligned} &-(2n-5)\gamma z^3 + (2n-5)3a\gamma \left| \begin{array}{c} z^2 \\ - (2n-4)\beta \end{array} \right| \begin{array}{c} + (2n-4)2a\beta \\ + 3a\gamma \end{array} \left| \begin{array}{c} - (2n-5)3a^2\gamma \\ - 6a^2\gamma \end{array} \right| \begin{array}{c} z + (2n-5)a^3\gamma \\ + 3a^3\gamma \end{array} \\ &= \frac{- (2n-4)a^2\beta \left| \begin{array}{c} - (2n-3)\alpha \\ + 2a\beta \end{array} \right| \begin{array}{c} + (2n-3)a\alpha \\ - 2a^2\beta \end{array} \left| \begin{array}{c} + a\alpha \end{array} \right|}{(x^2+a)^n} \cdot \frac{dx}{R}; \end{aligned}$$

or, after resubstituting  $z = x^2 + a$ , and integrating,

$$\begin{aligned}
 (15) \quad \frac{xR}{(x^2 + a)^{n-1}} &= -(2n-5)\gamma \int \frac{dx}{(x^2 + a)^{n-3}R} \\
 &\quad - (2n-4)(\beta - 3a\gamma) \int \frac{dx}{(x^2 + a)^{n-2}R} \\
 &\quad - (2n-3)(3a^2\gamma - 2a\beta + \alpha) \int \frac{dx}{(x^2 + a)^{n-1}R} \\
 &\quad + (2n-2)(a^3\gamma - a^2\beta + a\alpha) \int \frac{dx}{(x^2 + a)^n R} \\
 &= \alpha_1 \int \frac{dx}{(x^2 + a)^{n-3}R} + \beta_1 \int \frac{dx}{(x^2 + a)^{n-2}R} + \gamma_1 \int \frac{dx}{(x^2 + a)^{n-1}R} \\
 &\quad + \delta_1 \int \frac{dx}{(x^2 + a)^n R}.
 \end{aligned}$$

Making  $n = 2$ , we have

$$\begin{aligned}
 (16) \quad \frac{xR}{(x^2 + a)^{-1}} &= \alpha_1 \int \frac{(x^2 + a)dx}{R} + \beta_1 \int \frac{dx}{R} + \gamma_1 \int \frac{dx}{(x^2 + a)R} \\
 &\quad + \delta_1 \int \frac{dx}{(x^2 + a)^2 R}.
 \end{aligned}$$

Equation (16) shows that

$$\int \frac{dx}{(x^2 + a)^2 R}$$

depends upon the three forms

$$\int \frac{x^2 dx}{R}, \quad \int \frac{dx}{R}, \quad \text{and} \quad \int \frac{dx}{(x^2 + a)R},$$

the three types of equation (2), and equation (15) shows that the general form

$$\int \frac{dx}{(x^2 + a)^n R}$$

depends ultimately upon the same three types.

We have now discussed every form which the general equation (1) can assume, and shown that they all depend ultimately upon one or more of the three types contained in equation (2).

These three types are called the three Elliptic Integrals of the first, second, and third kind, respectively.

Legendre puts  $x = \sin \phi$ , and reduces the three integrals to the following forms:

$$(17) \quad F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}};$$

$$\frac{1}{k^2} \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} - \frac{1}{k^2} \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} \cdot d\phi;$$

$$(18) \quad \Pi(n, k, \phi) = \int_0^\phi \frac{d\phi}{(1 - n \sin^2 \phi) \sqrt{1 - k^2 \sin^2 \phi}};$$

the first being Legendre's integral of the first kind; the form

$$(19) \quad E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} \cdot d\phi$$

being the integral of the second kind; and the third one being the integral of the third kind.

The form of the integral of the second kind shows why they are called Elliptic Integrals, the arc of an elliptic quadrant being equal to

$$a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \phi} \cdot d\phi,$$

$\phi$  being the complement of the eccentric angle.

By easy substitutions, we get from Eqs. (17), (18), and (19) the following solutions:

$$\int_0^\phi \frac{\sin^2 \phi}{\Delta} d\phi = \frac{F - E}{k^2};$$

$$\int_0^\phi \frac{\cos^2 \phi}{\Delta} d\phi = \frac{E - (1 - k^2)F}{k^2};$$

$$\int_0^\phi \frac{\tan^2 \phi}{\Delta} d\phi = \frac{\Delta \tan \phi - E}{1 - k^2};$$

$$\int_0^\phi \frac{\sec^2 \phi}{\Delta} d\phi = \frac{\Delta \tan \phi + (1 - k^2)F - E}{1 - k^2};$$

$$\int_0^\phi \frac{1}{\Delta^3} d\phi = \frac{1}{1 - k^2} \left( E - \frac{k^2 \sin \phi \cos \phi}{\Delta} \right);$$

$$\int_0^\phi \frac{\sin^2 \phi}{\Delta^3} d\phi = \frac{1}{1 - k^2} \left( \frac{E - (1 - k^2)F}{k^2} - \frac{\sin \phi \cos \phi}{\Delta} \right);$$

$$\int_0^\phi \frac{\cos^2 \phi}{\Delta^3} d\phi = \frac{F - E}{k^2} + \frac{\sin \phi \cos \phi}{\Delta}.$$

## CHAPTER II.

### ELLIPTIC FUNCTIONS.

LET 
$$u = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

$\phi^*$  is called the *amplitude* corresponding to the *argument*  $u$ , and is written

$$\phi = \text{am}(u, k) = \text{am } u.$$

The quantity  $k$  is called the *modulus*, and the expression  $\sqrt{1 - k^2 \sin^2 \phi}$  is written \*

$$\sqrt{1 - k^2 \sin^2 \phi} = \Delta \text{ am } u = \Delta \phi,$$

and is called the *delta function* of the amplitude or  $u$ , or *delta of  $\phi$* , or simply *delta  $\phi$* .

$u$  can be written

$$u = F(k, \phi).$$

The following abbreviations are used:

$$\sin \phi = \sin \text{ am } u = \text{sn } \dagger u;$$

$$\cos \phi = \cos \text{ am } u = \text{cn } \dagger u;$$

$$\Delta \phi = \Delta \text{ am } u = \text{dn } \dagger u = \Delta u;$$

$$\tan \phi = \tan \text{ am } u = \text{tn } u.$$

Let  $\phi$  and  $\psi$  be any two arbitrary angles, and put

$$\phi = \text{am } u;$$

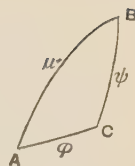
$$\psi = \text{am } v.$$

\* Legendre.

† Gudermann, in his "Theorie der Modularfunctionen": Crelle's Journal, Bd. 18.

In the spherical triangle  $ABC$  we have from Trigonometry,  $c$  and  $C$  being constant,

$$\frac{d\phi}{\cos B} + \frac{d\psi}{\cos A} = 0.$$



Since  $C$  and  $c$  are constant, denoting by  $k$  an arbitrary constant, we have

$$(1) \quad \frac{\sin C}{\sin \mu} = k.$$

But

$$\sin A = \sin \psi \frac{\sin B}{\sin \phi} = \sin \psi \frac{\sin C}{\sin \mu} = k \sin \psi.$$

Whence

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - k^2 \sin^2 \psi}.$$

In the same manner

$$\cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - k^2 \sin^2 \phi}.$$

Substituting these values, we get

$$(2) \quad \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} + \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = 0.$$

Integrating this, there results

$$(3) \quad \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} + \int_0^\psi \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \text{const.}$$

When  $\phi = 0$ , we have  $\psi = \mu$ , and therefore the constant must be of the form

$$\int_0^\mu \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

whence

$$(4) \int_0^\phi \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} + \int_0^\psi \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \int_0^\mu \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}},$$

or

$$u + v = m;$$

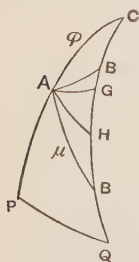
and evidently the amplitudes  $\phi$ ,  $\psi$ , and  $\mu$  can be considered as the three sides of a spherical triangle, and the relations between the sides of this spherical triangle will be the same as those between  $\phi$ ,  $\psi$ , and  $\mu$ .

But the sides of this triangle have imposed upon them the condition

$$\frac{\sin C}{\sin \mu} = k;$$

and since  $k < 1$ , we must have  $\mu > C$ , which requires that one of the angles of the triangle shall be obtuse and the other two acute.

In the figure, let  $C$  be an acute angle of the triangle  $ABC$ , and  $PQ$  the equatorial great circle of which  $C$  is the pole.



The arc  $PQ$  will be the measure of the angle  $C$ .

Let  $AG$  and  $AH$  be the arcs of two great circles perpendicular respectively to  $CQ$  and  $CP$ . They will of course be shorter than  $PQ$ . Hence  $AB = \mu$  must intersect  $CQ$  in points between  $CG$  and  $HQ$ , since  $\mu > (C = PQ)$ . In any case either  $A$  or  $B$  will be obtuse according as  $B$  falls between  $QH$  or  $CG$  respectively; and the other angle will be acute.

In the case where  $C$  is an obtuse angle, it will be easily seen that the angle at  $A$  must be acute, since the great circle  $AD$ , perpendicular to  $CP$ , intersects  $PQ$  in  $D$ ,  $PD$  being a quadrant. The same remarks apply to the angle  $B$ . Hence, in either

case, one of the angles of the triangle is obtuse and the other two are acute, as a result of the condition

$$\frac{\sin C}{\sin \mu} = k < 1.$$

From Trigonometry we have

$$\cos \mu = \cos \phi \cos \psi + \sin \phi \sin \psi \cos C;$$

and since the angle  $C$  is obtuse,

$$\cos C = -\sqrt{1 - \sin^2 C} = -\sqrt{1 - k^2 \sin^2 \mu},$$

and

$$(5) \quad \cos \mu = \cos \phi \cos \psi - \sin \phi \sin \psi \sqrt{1 - k^2 \sin^2 \mu},$$

the relation sought.

The spherical triangle likewise gives the following relations between the sides:

$$(5)^* \quad \begin{cases} \cos \phi = \cos \mu \cos \psi + \sin \mu \sin \psi \sqrt{1 - k^2 \sin^2 \phi}; \\ \cos \psi = \cos \mu \cos \phi + \sin \mu \sin \phi \sqrt{1 - k^2 \sin^2 \psi}. \end{cases}$$

These give, by eliminating  $\cos \mu$ ,

$$\sin \mu = \frac{\cos^2 \psi - \cos^2 \phi}{\sin \phi \cos \psi \Delta \psi - \sin \psi \cos \phi \Delta \phi};$$

which, after multiplying by the sum of the terms in the denominator and substituting  $\cos^2 = 1 - \sin^2$ , can be written

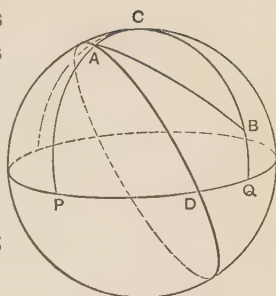
$$\sin \mu = \frac{(\sin^2 \phi - \sin^2 \psi)(\sin \phi \cos \psi \Delta \psi + \sin \psi \cos \phi \Delta \phi)}{\sin^2 \phi \cos^2 \psi \Delta^2 \psi - \sin^2 \psi \cos^2 \phi \Delta^2 \phi}.$$

Since the denominator can be written

$$(6) \quad \sin \mu = \frac{(\sin^2 \phi - \sin^2 \psi)(1 - k^2 \sin^2 \phi \sin^2 \psi),}{1 - k^2 \sin^2 \phi \sin^2 \psi}.$$

In a similar manner we get

$$(6)^* \quad \begin{cases} \cos \mu = \frac{\cos \phi \cos \psi - \sin \phi \sin \psi \Delta \phi \Delta \psi}{1 - k^2 \sin^2 \phi \sin^2 \psi}; \\ \Delta \mu = \frac{\Delta \phi \Delta \psi - k^2 \sin \phi \sin \psi \cos \phi \cos \psi}{1 - k^2 \sin^2 \phi \sin^2 \psi}. \end{cases}$$



These equations can also be written as follows :

$$(7) \begin{cases} \sin \operatorname{am} (u \pm v) = \frac{\sin \operatorname{am} u \cos \operatorname{am} v \Delta \operatorname{am} v \pm \sin \operatorname{am} v \cos \operatorname{am} u \Delta \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v}; \\ \cos \operatorname{am} (u \pm v) = \frac{\cos \operatorname{am} u \cos \operatorname{am} v \mp \sin \operatorname{am} u \sin \operatorname{am} v \Delta \operatorname{am} u \Delta \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v}; \\ \Delta \operatorname{am} (u \pm v) = \frac{\Delta \operatorname{am} u \Delta \operatorname{am} v \mp k^2 \sin \operatorname{am} u \sin \operatorname{am} v \cos \operatorname{am} u \cos \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v}; \end{cases}$$

or

$$(8) \begin{cases} \operatorname{sn} (u \pm v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}; \\ \operatorname{cn} (u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}; \\ \operatorname{dn} (u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \end{cases}$$

Making  $u = v$ , we get from the upper sign

$$(9) \begin{cases} \operatorname{sn} 2u = \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u}; \\ \operatorname{cn} 2u = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \frac{1 - 2 \operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1 - k^2 \operatorname{sn}^4 u}; \\ \operatorname{dn} 2u = \frac{\operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u} = \frac{1 - 2k^2 \operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1 - k^2 \operatorname{sn}^4 u}. \end{cases}$$

From these

$$(10) \begin{cases} 1 - \operatorname{cn} 2u = \frac{2 \operatorname{cn}^2 u \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u}; \\ 1 + \operatorname{cn} 2u = \frac{2 \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u}; \\ 1 - \operatorname{dn} u = \frac{2k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u}; \\ 1 + \operatorname{dn} u = \frac{2 \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u}; \end{cases}$$

and therefore

$$(11) \quad \begin{cases} \operatorname{sn}^2 u = \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}; \\ \operatorname{cn}^2 u = \frac{\operatorname{dn} 2u + \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}; \\ \operatorname{dn}^2 u = \frac{1 - k^2 + \operatorname{dn} 2u + k^2 \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}; \end{cases}$$

and by analogy

$$(12) \quad \begin{cases} \operatorname{sn} \frac{u}{2} = \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}; \\ \operatorname{cn} \frac{u}{2} = \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}}; \\ \operatorname{dn} \frac{u}{2} = \sqrt{\frac{1 - k^2 + \operatorname{dn} u + k^2 \operatorname{cn} u}{1 + \operatorname{dn} u}}. \end{cases}$$

In equations (7) making  $u = v$ , and taking the lower sign, we have

$$(13) \quad \begin{cases} \operatorname{sn} 0 = 0; \\ \operatorname{cn} 0 = 1; \\ \operatorname{dn} 0 = 1. \end{cases}$$

Likewise, we get by making  $u = 0$ ,

$$(14) \quad \begin{cases} \operatorname{sn} (-u) = -\operatorname{sn} u; \\ \operatorname{cn} (-u) = +\operatorname{cn} u; \\ \operatorname{dn} (-u) = \operatorname{dn} u. \end{cases}$$

### CHAPTER III.

#### PERIODICITY OF THE FUNCTIONS.

WHEN the elliptic integral

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

has for its amplitude  $\frac{\pi}{2}$ , it is called, following the notation of Legendre, the *complete* function, and is indicated by  $K$ , thus:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

When  $k$  becomes the complementary modulus,  $k'$ , (see eq. 4, Chap. IV,) the corresponding complete function is indicated by  $K'$ , thus:

$$K' = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k'^2 \sin^2 \phi}}.$$

From these, evidently,

$$\text{am } (K, k) = \frac{\pi}{2}, \quad \text{am } (K', k') = \frac{\pi}{2}.$$

$$(1) \quad \begin{cases} \text{sn } (K, k) = 1; \\ \text{cn } (K, k) = 0; \\ \text{dn } (K, k) = k'. \end{cases}$$

From eqs. (7), (8), and (9), Chap. II, we have, by the substitution of the values of  $\text{sn } (K) = 1$ ,  $\text{cn } (K) = 0$ ,  $\text{dn } (K) = k'$ ,

$$(2) \quad \begin{cases} \text{sn } 2K = 0; \\ \text{cn } 2K = -1; \\ \text{dn } 2K = 1. \end{cases}$$

These equations, by means of (1), (2), and (3) of Chap. II, give

$$(3) \quad \begin{cases} \text{sn } (u + 2K) = -\text{sn } u; \\ \text{cn } (u + 2K) = -\text{cn } u; \\ \text{dn } (u + 2K) = \text{dn } u; \end{cases}$$

and these, by changing  $u$  into  $u + 2K$ , give

$$(4) \quad \begin{cases} \text{sn } (u + 4K) = \text{sn } u; \\ \text{cn } (u + 4K) = \text{cn } u; \\ \text{dn } (u + 4K) = \text{dn } u. \end{cases}$$

From these equations it is seen that the elliptic functions  $\text{sn}$ ,  $\text{cn}$ ,  $\text{dn}$ , are periodic functions having for their period  $4K$ . Unlike the period of trigonometric functions, this period is not a fixed one, but depends upon the value of  $k$ , the modulus.

From the Integral Calculus we have

$$\begin{aligned} \int_0^{n\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} &= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\phi}{\Delta\phi} + \int_{\pi}^{\frac{3\pi}{2}} \frac{d\phi}{\Delta\phi} + \dots + \int_{(n-1)\frac{\pi}{2}}^{n\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} \\ &= n \int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} = nK; \end{aligned}$$

from which we see that

$$n\frac{\pi}{2} = \text{am } (nK);$$

or, since  $\frac{\pi}{2} = \text{am } K$ ,

$$\text{am } (nK) = n \cdot \text{am } K,$$

and

$$n\pi = \text{am } (2nK),$$

and also

$$n\pi = 2n \text{ am } K.$$

In the case of an Elliptic Integral with the arbitrary angle  $\alpha$ , we can put

$$d = n\pi \pm \beta,$$

where  $\beta$  is an angle between 0 and  $\frac{\pi}{2}$ , the upper or the lower sign being taken according as  $\frac{\pi}{2}$  is contained in  $\alpha$  an even or an uneven number of times.

In the first case we have

$$\int_0^{n\pi + \beta} \frac{d\phi}{\Delta\phi} = \int_0^{n\pi} \frac{d\phi}{\Delta\phi} + \int_{n\pi}^{n\pi + \beta} \frac{d\phi}{\Delta\phi};$$

or, putting  $\phi_1 = -n\pi$ ,

$$\int_0^{n\pi + \beta} \frac{d\phi}{\Delta\phi} = 2nK + \int_0^{\beta} \frac{d\phi_1}{\Delta\phi_1}.$$

In the second case

$$\int_0^{n\pi - \beta} \frac{d\phi}{\Delta\phi} = \int_0^{n\pi} \frac{d\phi}{\Delta\phi} - \int_{n\pi - \beta}^{n\pi} \frac{d\phi}{\Delta\phi};$$

or, putting  $\phi_1 = n\pi - \phi$ ,

$$\int_0^{n\pi - \beta} \frac{d\phi}{\Delta\phi} = 2nK - \int_0^{\beta} \frac{d\phi_1}{\Delta\phi_1};$$

or in either case,

$$\int_0^{n\pi \pm \beta} \frac{d\phi}{\Delta\phi} = 2nK \pm \int_0^{\beta} \frac{d\phi_1}{\Delta\phi_1}.$$

Thus we see that the Integral with the general amplitude  $\alpha$  can be made to depend upon the complete integral  $K$  and an Integral whose amplitude lies between 0 and  $\frac{\pi}{2}$ .

Put now

$$\int_0^{\beta} \frac{d\phi_1}{\Delta\phi_1} = u, \quad \beta = \text{am } u.$$

This gives

$$\int_0^{n\pi \pm \beta} \frac{d\phi}{\Delta\phi} = 2nK \pm u,$$

or

$$\text{am } (2nK \pm u) = n\pi \pm \beta$$

(5)

$$= n\pi \pm \text{am } u$$

(6)

$$= 2n \cdot \text{am } K \pm \text{am } u;$$

or, since

$$\text{am } (-z) = -\text{am } z,$$

$$\text{am } (u \pm 2nK) = \text{am } u \pm n\pi$$

$$= \text{am } u \pm 2n \cdot \text{am } K.$$

Taking the sine and cosine of both sides, we have

$$\text{sn } (u \pm 2nK) = \pm \text{sn } u;$$

$$\text{cn } (u \pm 2nK) = \pm \text{cn } u;$$

the upper or the lower sign being taken according as  $n$  is even or odd. By giving the proper values to  $n$  we can get the same results as in equations (3) and (4).

Putting  $n = 1$  in eq. (5), we have

$$\begin{aligned} \text{sn } (2K - u) &= \sin \pi \text{cn } u - \cos \pi \text{sn } u \\ (7) \quad &= \text{sn } u. \end{aligned}$$

Elliptic functions also have an imaginary period. In order to show this we will, in the integral

$$\int_0^\phi \frac{d\phi}{\Delta\phi},$$

assume the amplitude as imaginary. Put

$$\sin \phi = i \tan \psi.$$

From this we get

$$(8) \quad \begin{cases} \cos \phi = \frac{1}{\cos \psi}; \\ \Delta\phi = \frac{\sqrt{1 - k'^2 \sin^2 \psi}}{\cos \psi} = \frac{\Delta(\psi, k')}{\cos \psi}; \\ d\phi = i \frac{d\psi}{\cos \psi}. \end{cases}$$

From these, since  $\phi$  and  $\psi$  vanish simultaneously, we easily get

$$\int_0^\phi \frac{d\phi}{\Delta\phi} = i \int_0^\psi \frac{d\psi}{\Delta(\psi, k')}.$$

Put

$$\int_0^\psi \frac{d\psi}{\Delta(\psi, k')} = u \quad \text{and} \quad \psi = \text{am}(u, k'),$$

whence

$$\int_0^\phi \frac{d\phi}{\Delta\phi} = iu \quad \text{and} \quad \phi = \text{am}(iu);$$

and these substituted in Eq. (8) give

$$(9) \quad \begin{cases} \text{sn } iu = i \text{tn}(u, k'); \\ \text{cn } iu = \frac{1}{\text{cn}(u, k')}; \\ \text{dn } iu = \frac{\text{dn}(u, k')}{\text{cn}(u, k')}. \end{cases}$$

By assuming

$$\int_0^\psi \frac{d\psi}{\Delta(\psi, k')} = iu \quad \text{and} \quad \int_0^\phi \frac{d\phi}{\Delta\phi} = -u,$$

we get

$$\operatorname{sn}(-u) = i \operatorname{tn}(iu, k'),$$

$$\operatorname{cn}(-u) = \frac{1}{\operatorname{cn}(iu, k')},$$

$$\operatorname{dn}(-u) = \frac{\operatorname{dn}(iu, k')}{\operatorname{cn}(iu, k')};$$

or, from eq. (14), Chap. II,

$$(10) \quad \begin{cases} \operatorname{sn} u = -i \operatorname{tn}(iu, k'); \\ \operatorname{cn} u = \frac{1}{\operatorname{cn}(iu, k')}; \\ \operatorname{dn} u = \frac{\operatorname{dn}(iu, k')}{\operatorname{cn}(iu, k')}. \end{cases}$$

From eqs. (7), Chap. II, making  $\nu = K$ , we get, since  $\operatorname{sn} K = 1$ ,  $\operatorname{cn} K = 0$ ,  $\operatorname{dn} K = k'$ ,

$$(11) \quad \begin{cases} \operatorname{sn}(u \pm K) = \pm \frac{\operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u} = \pm \frac{\operatorname{cn} u}{\operatorname{dn} u}; \\ \operatorname{cn}(u \pm K) = \frac{\mp \operatorname{sn} u \operatorname{dn} u k'}{\operatorname{dn}^2 u} = \mp \frac{k' \operatorname{sn} u}{\operatorname{dn} u}; \\ \operatorname{dn}(u \pm K) = \pm \frac{k'}{\operatorname{dn} u}. \end{cases}$$

In these equations, changing  $u$  into  $iu$ , we get, by means of eqs. (9),

$$(12) \quad \begin{cases} \operatorname{sn}(iu \pm K) = \pm \frac{1}{\operatorname{dn}(u, k')}; \\ \operatorname{cn}(iu \pm K) = \mp \frac{ik' \operatorname{sn}(u, k')}{\operatorname{dn}(u, k')}; \\ \operatorname{dn}(iu \pm K) = \pm \frac{k' \operatorname{cn}(u, k')}{\operatorname{dn}(u, k')}. \end{cases}$$

Putting now in eqs. (9)  $u \pm K'$  instead of  $u$ , and making use of eqs. (10), and interchanging  $k$  and  $k'$ , we have

$$(13) \quad \begin{cases} \operatorname{sn}(iu \pm iK') = -\frac{i \operatorname{cn}(u, k')}{k \operatorname{sn}(u, k')}; \\ \operatorname{cn}(iu \pm iK') = \mp \frac{\operatorname{dn}(u, k')}{k \operatorname{sn}(u, k')}; \\ \operatorname{dn}(iu \pm iK') = \mp \frac{1}{\operatorname{sn}(u, k')}. \end{cases}$$

Substituting in these  $-iu$  in place of  $u$ , we get, by means of eqs. (9) and eqs. (14) of Chap. II,

$$(14) \quad \begin{cases} \operatorname{sn}(u \pm iK') = \frac{1}{k \operatorname{sn} u}; \\ \operatorname{cn}(u \pm iK') = \mp \frac{i \operatorname{dn} u}{k \operatorname{sn} u}; \\ \operatorname{dn}(u \pm iK') = \mp i \cot \operatorname{am} u. \end{cases}$$

In these equations, putting  $u + K$  in place of  $u$ , we get

$$(15) \quad \begin{cases} \operatorname{sn}(u + K \pm iK') = \pm \frac{\operatorname{dn} u}{k \operatorname{cn} u}; \\ \operatorname{cn}(u + K \pm iK') = \mp \frac{ik'}{k \operatorname{cn} u}; \\ \operatorname{dn}(u + K \pm iK') = \pm ik' \operatorname{tn} u. \end{cases}$$

Whence for  $u = 0$  we get

$$(16) \quad \begin{cases} \operatorname{sn}(K \pm iK') = \frac{1}{k}; \\ \operatorname{cn}(K \pm iK') = \mp \frac{ik'}{k}; \\ \operatorname{dn}(K \pm iK') = 0. \end{cases}$$

If in eqs. (14) we put  $u = 0$ , we see that as  $u$  approaches zero, the expressions

$$\operatorname{sn}(\pm iK'), \quad \operatorname{cn}(\pm iK'), \quad \operatorname{dn}(\pm iK')$$

approach infinity.

We see from what has preceded that Elliptic Functions have two periods, one a real period, and one an imaginary period.

In the former characteristic they resemble Trigonometric Functions, and in the latter Logarithmic Functions.

On account of these two periods they are often called Doubly Periodic Functions. Some authors make this double periodicity the starting-point of their investigations. This method of investigation gives some very beautiful results and processes, but not of a kind adapted for an elementary work.

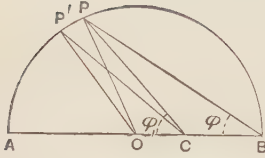
It will be noticed that the Elliptic Functions  $\operatorname{sn} u$ ,  $\operatorname{cn} u$ , and  $\operatorname{dn} u$  have a very close analogy to trigonometric functions, in which, however, the independent variable  $u$  is not an angle, as it is in the case of trigonometric functions.

Like Trigonometric Functions, these Elliptic Functions can be arranged in tables. These tables, however, require a double argument, viz.,  $u$  and  $k$ . In Chap. IX these functions are developed into series, from which their values may be computed and tables formed.

No complete tables have yet been published, though they are in process of computation.

## CHAPTER IV.

### LANDEN'S TRANSFORMATION.



LET  $AB$  be the diameter of a circle, with the centre at  $O$ , the radius  $AO = r$ , and  $C$  a fixed point situated upon  $OB$ , and  $OC = k_0 r$ . Denote the angle  $PBC$  by  $\phi$ , and the angle  $PCO$  by  $\phi_1$ . Let  $P'$  be a point indefinitely near to  $P$ .

Then

$$\frac{PP'}{PC} = \frac{\sin PCP'}{\sin PP'C} = \frac{\sin PCP'}{\cos OP'C}$$

But  $PP' = 2rd\phi$ , and  $\sin PCP' = PCP' = d\phi_1$ ;  
therefore

$$\frac{2rd\phi}{PC} = \frac{d\phi_1}{\cos OP'C}$$

But

$$\begin{aligned} \overline{PC}^2 &= r^2 + r^2 k_0^2 + 2r^2 k_0 \cos 2\phi \\ &= (r + rk_0)^2 \cos^2 \phi + (r - rk_0)^2 \sin^2 \phi; \\ \text{also } r^2 \cos^2 OP'C &= r^2 - r^2 \sin^2 OP'C \\ &= r^2 - r^2 k_0^2 \sin^2 \phi_1. \end{aligned}$$

Therefore

$$\frac{2d\phi}{\sqrt{(r + rk_0)^2 \cos^2 \phi + (r - rk_0)^2 \sin^2 \phi}} = \frac{d\phi_1}{\sqrt{r^2 - r^2 k_0^2 \sin^2 \phi_1}},$$

which can be written

$$\frac{2}{r + rk_0} \frac{d\phi}{\sqrt{1 - \frac{4k_0^2 r^2}{(r + rk_0)^2} \sin^2 \phi}} = \frac{1}{r} \frac{d\phi_1}{\sqrt{1 - k_0^2 \sin^2 \phi_1}},$$

Putting

$$(1) \quad \frac{4k_0 r^2}{(r + rk_0)^2} = \frac{4k_0}{(1 + k_0)^2} = k^2,$$

we have

$$(2) \quad \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{1 + k_0}{2} \int_0^{\phi_1} \frac{d\phi_1}{\sqrt{1 - k_0^2 \sin^2 \phi_1}};$$

no constant being added because  $\phi$  and  $\phi_1$  vanish simultaneously;  $\phi$  and  $\phi_1$  being connected by the equation

$$(3) \quad \frac{\sin OPC}{\sin OCP} = \frac{\sin (2\phi - \phi_1)}{\sin \phi_1} = \frac{rk_0}{r} = k_0.$$

From the value of  $k^2$  we have

$$(4) \quad 1 - k^2 = k'^2 = \frac{(1 - k_0)^2}{(1 + k_0)^2},$$

and therefore

$$(5) \quad k_0 = \frac{1 - k'}{1 + k'}.$$

$k'$  is called the *complementary modulus*, and is evidently the minimum value of  $\Delta\phi$ , the value of  $\Delta\phi$  when  $\phi = 90^\circ$ :

$$\sqrt{1 - k^2} = k'.$$

From eq. (1) we evidently have  $k > k_0$ , for, putting eq. (1) in the form

$$\frac{k^2}{k_0^2} = \frac{4}{k_0 + 2k_0^2 + k_0^3},$$

we see that if  $k_0 = 1$ , then  $k = k_0$ , but as  $k_0 < 1$ , always, as is evident from the figure,  $k$  must be greater than  $k_0$ .

It is also evident, from the figure, that  $\phi_1 > \phi$ . Or it may be deduced directly from eq. (3).

Since  $k < 1$ , we can write

$$k = \sin \theta, \quad k' = \sqrt{1 - k^2} = \cos \theta.$$

Substituting in eq. (5), we have

$$k_0 = \frac{1 - k'}{1 + k'} = \tan^2 \frac{1}{2}\theta,$$

and we can write

$$k_0 = \sin \theta_0, \quad k_1' = \sqrt{1 - k_0^2} = \cos \theta_0.$$

From eq. (5) we have

$$1 + k_0 = \frac{2}{1 + k'}.$$

Substituting the value of  $k_0$  in that for  $k_1'$ , we get

$$k_1' = \frac{2\sqrt{k'}}{1 + k'}.$$

We also have

$$2\phi - \phi_1 = \phi - (\phi_1 - \phi)$$

$$\phi_1 = \phi + (\phi_1 - \phi),$$

and, eq. (3),

$$\operatorname{sn}(2\phi - \phi_1) = k_0 \sin \phi_1,$$

becomes

$$\sin \varphi \cos(\phi_1 - \phi) - \cos \phi \sin(\phi_1 - \phi) =$$

$$k_0 \sin \phi \cos(\phi_1 - \phi) + k_0 \cos \phi \sin(\phi_1 - \phi),$$

or

$$\tan \phi - \tan(\phi_1 - \phi) = k_0 \tan \phi + k_0 \tan(\phi_1 - \phi),$$

or

$$\tan(\phi_1 - \phi) = \frac{1 - k_0}{1 + k_0} \tan \phi$$

$$= k' \tan \phi.$$

Collecting these results, we have

$$(6) \quad k = \frac{2\sqrt{k_0}}{1 + k_0} = \sin \theta;$$

$$(7) \quad k_0 = \frac{1 - k'}{1 + k'} = \sin \theta_0 = \tan^2 \frac{1}{2}\theta;$$

$$(8) \quad k_1' = \frac{2\sqrt{k'}}{1+k'} = \cos \theta_0;$$

$$(9) \quad k' = \frac{1-k_0}{1+k_0} = \cos \theta;$$

$$(10) \quad 1+k_0 = \frac{2}{1+k'} = \frac{2\sqrt{k_0}}{k} = \frac{k_1'}{\sqrt{k'}} = \frac{1}{\cos^2 \frac{1}{2}\theta};$$

$$(11) \quad \sin (2\phi - \phi_1) = k_0 \sin \phi_1;$$

$$(12) \quad \tan (\phi_1 - \phi) = k' \tan \phi;$$

$$(13) \quad \int_0^\phi \frac{d\phi}{\Delta(k, \phi)} = \frac{1+k_0}{2} \int_0^{\phi_1} \frac{d\phi_1}{\Delta(k_0, \phi_1)};$$

$$(14) \quad k = \sqrt{1-k'^2}, \quad k' = \sqrt{1-k^2}.$$

Upon examination it will easily appear that  $k$  and  $k_0$ , and  $\theta$  and  $\theta_0$ , are the first two terms of a decreasing series of moduli and angles;  $k'$  and  $k_1'$ , and  $\phi$  and  $\phi_1$ , of an increasing series; the law connecting the different terms of the series being deduced from eqs. (6) to (12).

By repeated applications of these equations we would get the following series of moduli and amplitudes:

|                           |                         |          |
|---------------------------|-------------------------|----------|
| $k_{0n} = O_{(n=\infty)}$ | $k_n' = I_{(n=\infty)}$ | $\phi_n$ |
| $\vdots$                  | $\vdots$                | $\vdots$ |
| $\vdots$                  | $\vdots$                | $\vdots$ |
| $k_{00}$                  | $k_2'$                  | $\phi_2$ |
| $k_0$                     | $k_1'$                  | $\phi_1$ |
| $k$                       | $k'$                    | $\phi$   |

The upper limit of the one series of moduli is 1, and the lower limit of the other series is 0, as is indicated.  $k$  and  $k'$ ,

which are bound by the relation  $k^2 + k'^2 = 1$ , are called the *primitives* of the series.

NOTE.—It will be noticed that the successive terms of a decreasing series are indicated by the sub-accents 0, 00, 03, 04, . . . 0*n*; and the successive terms of an increasing series by the sub-accents 1, 2, 3, . . . *n*.

Again, by application of these equations, we can form a new series running up from  $k$ , viz.,  $k_1, k_2, k_3, \dots k_n = 1_{(n=\infty)}$ ; and also a new series running down from  $k'$ , viz.,  $k'_0, k'_{00}, \dots k'_{0n} = 0_{(n=\infty)}$ . So also with  $\phi$ .

Collecting these series, we have

|                       |                        |                 |
|-----------------------|------------------------|-----------------|
| $k_{0n} = 0$          | $k'_n = 1$             | $\phi_n$        |
| ⋮                     | ⋮                      | ⋮               |
| $k_{02}$              | $k'_2$                 | $\phi_2$        |
| $k_0$                 | $k'_1$                 | $\phi_1$        |
| $k \dots \dots \dots$ | $k' \dots \dots \dots$ | $\phi$          |
| $k_1$                 | $k'_0$                 | $\phi_0$        |
| $k_2$                 | $k'_{00}$              | $\phi_{00}$     |
| ⋮                     | ⋮                      | ⋮               |
| ⋮                     | ⋮                      | ⋮               |
| $k_n = 1$             | $k'_{0n} = 0$          | $\phi_{0n} = 0$ |

NOTE.—In practice it will be found that generally *n* will not need to be very large in order to reach the limiting values of the terms, often only two or three terms being needed.

Applying eqs. (7), (12), (13), and (14) repeatedly, we get

$$(14.) \left\{ \begin{array}{ll} k = \sin \theta, & k' = \cos \theta; \\ k_0 = \frac{1 - k'}{1 + k'} = \tan^2 \frac{1}{2} \theta = \sin \theta_0, & k'_1 = \cos \theta_0; \\ k_{00} = \tan^2 \frac{1}{2} \theta_0 = \sin \theta_{00}, & k'_2 = \cos \theta_{00}; \\ k_{03} = \tan^2 \frac{1}{2} \theta_{00} = \sin \theta_{03}, & k'_3 = \cos \theta_{03}; \\ \dots \dots \dots & \dots \dots \dots \\ k_{0n} = \tan^2 \frac{1}{2} \theta_{0(n-1)} = \sin \theta_{0n}, & k'_n = \cos \theta_{0n}. \end{array} \right.$$



From eqs. (15) and (10) we get

$$F(k, \phi) = \sqrt{\frac{k'_1 k'_2 k'_3 \dots k'_n}{k'}} \cdot \frac{F(k_{0n}, \phi_n)}{2^n}.$$

And this with equations (8) and (9) gives

$$(17) \quad F(k, \phi) = \sqrt{\frac{\cos \theta_0 \cos \theta_{00} \dots \cos^2 \theta_{0n}}{\cos \theta}} \cdot \frac{F(k_{0n}, \phi_n)}{2^n}.$$

Applying equation (13) to  $(k_1, \phi_0)$ ,  $(k_2, \phi_{00})$ , etc., we get

$$F(k_1, \phi_0) = \frac{1+k}{2} F(k, \phi), \text{ etc.};$$

but since, eq. (10),

$$\frac{1+k}{2} = \frac{1}{1+k'_0}, \text{ etc.},$$

these become

$$F(k, \phi) = (1+k'_0)F(k_1, \phi_0);$$

$$F(k_1, \phi_0) = (1+k'_{00})F(k_2, \phi_{00});$$

$$\dots \dots \dots$$

$$F(k_{n-1}, \phi_{0(n-1)}) = (1+k'_{0n})F(k_{0n}, \phi_{0n});$$

whence

$$(18) \quad F(k, \phi) = (1+k'_0)(1+k'_{00}) \dots (1+k'_{0n})F(k_n, \phi_{0n}),$$

in which  $k'_0$ ,  $k'_{00}$ , etc.,  $k_1$ ,  $k_2$ , etc.,  $\phi_0$ ,  $\phi_{00}$ , etc., are determined as follows:

$$\begin{array}{ll} \text{Let} & k = \sin \theta, \\ & k_1 = \sin \theta_1. \end{array}$$

From eq. (10),

$$k_1 = \frac{2\sqrt{k}}{1+k} \quad \text{or} \quad \sin \theta_1 = \frac{2\sqrt{\sin \theta}}{1+\sin \theta}.$$

Solving this equation for  $\sin \theta$ , we get

$$\sin \theta = \tan^2 \frac{1}{2} \theta_1.$$

Hence we can write

$$(18_1) \quad \begin{cases} k = \sin \theta = \tan^2 \frac{1}{2} \theta_1; \\ k_1 = \sin \theta_1 = \tan^2 \frac{1}{2} \theta_2; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ k_n = \sin \theta_n. \end{cases}$$

From equation (12) we get

$$(18_2) \quad \begin{cases} \sin (2\phi_0 - \phi) = k \sin \phi; * \\ \sin (2\phi_{00} - \phi_0) = k_1 \sin \phi_0; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sin (2\phi_{0n} - \phi_{0(n-1)}) = k_{n-1} \sin \phi_{0(n-1)}. \end{cases}$$

\* When  $\sin \phi = 1$  nearly,  $\phi$  is best determined as follows: From eq. (12) we have

$$\begin{aligned} \tan (\phi - \phi_0) &= k'_0 \tan \phi_0 \\ &= k'_0 \tan \phi \text{ nearly;} \end{aligned}$$

whence

$$\phi - \phi_0 = R k'_0 \tan \phi \text{ nearly,}$$

$R$  being the radian in seconds, viz. 206264''.806, and  $\log R = 5.3144251$ .

Substituting the approximate value of  $\phi_0$ , we can get a new approximation.

*Example.*  $\phi_0 = 82^\circ 30'$   $k'_{00} = \log^{-1} 5.8757219$

|                     |            |                       |
|---------------------|------------|-----------------------|
| $\tan 82^\circ 30'$ | 10.8805709 |                       |
| $k'_{00}$           | 5.8757219  |                       |
| $R$                 | 5.3144251  |                       |
|                     | <hr/>      |                       |
|                     | 2.0707179  | $117''.684 = 1'.9614$ |

$$\phi_0 - \phi_{00} = 1'.9614$$

$$\phi_{00} = 82^\circ 28'.0386$$

1st approximation.

To determine  $k'_0, k'_{00}$ , etc., we have

$$(18_s) \quad \begin{cases} k' = \sin \eta, & k = \cos \eta; \\ k'_0 = \frac{1-k}{1+k} = \tan^2 \frac{1}{2}\eta = \sin \eta_0, & k_1 = \cos \eta_0; \\ k'_{00} = \tan^2 \frac{1}{2}\eta_0 = \sin \eta_{00}, & k_2 = \cos \eta_{00}; \\ \text{etc.} & \text{etc.} & \text{etc.} \end{cases}$$

Or, since  $1 + k'_0 = \frac{1}{\cos^2 \frac{1}{2}\eta}$ ,  $1 + k'_{00} = \frac{1}{\cos^2 \frac{1}{2}\eta_0}$ , etc.,

we can put eq. (18) in the following form :

$$(19) \quad F(k, \phi) = \frac{1}{\cos^2 \frac{1}{2}\eta \cos^2 \frac{1}{2}\eta_0 \dots \cos^2 \frac{1}{2}\eta_{0n}} F(k_n, \phi_{0n}).$$

From equation (13) we have

$$(19)^* \quad F(k_1, \phi_0) = \frac{1+k}{2} F(k, \phi),$$

whence

$$F(k, \phi) = \frac{2}{1+k} F(k_1, \phi_0).$$

By repeated applications this gives, after combining,

$$F(k, \phi) = \frac{2}{1+k} \cdot \frac{2}{1+k_1} \dots \frac{2}{1+k_{n-1}} \cdot F(k_n, \phi_{0n})$$

This value gives

$$\phi_0 - \phi_{00} = 117''.1675 = 1'.95279$$

$$\therefore \phi_{00} = 82^\circ 28'.04721$$

2d approximation.

This value gives

$$\phi_0 - \phi_{00} = 117''.1698 = 1'.95283$$

$$\phi_{00} = 82^\circ 28'.04717$$

3d approximation.

$$= \frac{k_1}{\sqrt{k}} \cdot \frac{k_2}{\sqrt{k_1}} \cdots \frac{k_n}{\sqrt{k_{n-1}}} \cdot F(k_n, \phi_{0n});$$

$$(20) \quad F(k, \phi) = \sqrt{\frac{k_1 k_2 \cdots k_n^2}{k}} \cdot F(k_n, \phi_{0n});$$

$k_1, k_2$ , etc., being determined by repeated applications of

$$k_1 = \frac{2\sqrt{k}}{1+k},$$

or by equations (18<sub>1</sub>).

In equation (19)\* let us change  $k_1$  and  $\phi_0$  into  $k'$  and  $\phi$  respectively, so that the first member may have for its complete function

$$K' = F(k', \phi).$$

Upon examination of eq. (19)\* we see that the modulus in the second member must be the one next less than the one in the first member, that is,  $k'_0$ ; and likewise that the amplitude must be the one next greater than the amplitude in the first member, viz.,  $\phi_1$ ; hence we get

$$F(k', \phi) = \frac{1+k'_0}{2} F(k'_0, \phi_1).$$

Indicating the complete functions by  $K'$  and  $K'_0$ , we have, since  $\phi = \frac{\pi}{2}$  when  $\phi_1 = \pi$  (see Chap. V),

$$K' = (1+k'_0)K'_0;$$

and in the same manner,

$$\begin{aligned} K'_0 &= (1+k'_{00})K'_{00}, \\ K'_{00} &= (1+k'_{03})K'_{03}, \\ &\vdots \\ K'_{0(n-1)} &= (1+k'_{0n})K'_{0n}; \end{aligned}$$

whence

$$K' = (1 + k'_0)(1 + k'_{00}) \dots (1 + k'_{0n})K'_{0n}.$$

Since

$$K'_{0n} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2}, \quad (n = \text{limit,})$$

we have

$$(20)^* \quad (1 + k'_0)(1 + k'_{00}) \dots (1 + k'_{0n}) = \frac{2K'}{\pi}.$$

From eq. (19)\* we have, since [eq. (10), Chap IV]

$$\frac{1 + k}{2} = \frac{1}{1 + k'_0},$$

$$(1 + k'_0) \int_0^{\phi_0} \frac{d\phi_0}{\Delta(\phi_0, k_1)} = \int_0^{\phi} \frac{d\phi}{\Delta(\phi, k)};$$

whence also, since for  $\phi_0 = \frac{\pi}{2}$ ,  $\phi = \pi$ ,

$$\begin{aligned} (1 + k'_0) K_1 &= 2K, \\ (1 + k'_{00}) K_2 &= 2K_1, \\ &\dots \dots \dots \\ (1 + k'_{0n}) K_n &= 2K_{n-1}, \end{aligned}$$

and

$$(1 + k'_0)(1 + k'_{00}) \dots (1 + k'_{0n})K_n = 2^n K;$$

or

$$\frac{K_n}{2^n} = \frac{K}{(1 + k'_0)(1 + k'_{00}) \dots} \quad (n = \infty)$$

$$(21) \quad = \frac{\pi}{2K_1} K.$$

Let us find the limiting value of  $F(k_{0n}, \phi_n)$  in eq. (15). In the equation  $\tan(\phi_n - \phi_{n-1}) = k_{n-1} \tan \phi_{n-1}$ , we see that when  $k_{n-1}$  reaches the limit 1, then  $\phi_n - \phi_{n-1} = \phi_{n-1}$  or  $\phi_n = 2\phi_{n-1}$ . Therefore

$$\frac{\phi_n}{2^n} = \frac{2\phi_{n-1}}{2^n} = \frac{\phi_{n-1}}{2^{n-1}};$$

$$\frac{\phi_{n+1}}{2^{n+1}} = \frac{2\phi_n}{2^{n+1}} = \frac{\phi_n}{2^n} = \frac{\phi_{n-1}}{2^{n-1}};$$

$$\frac{\phi_{n+m}}{2^{n+m}} = \frac{\phi_{n-1}}{2^n} = \text{constant, whatever } m \text{ may be.}$$

Therefore eq. (15) becomes

$$(21)^* \quad F(k, \phi) = (1 + k_0)(1 + k_{00}) \dots (1 + k_{0n}) \frac{\phi_n}{2^n},$$

$n$  being whatever number will carry  $k_0$  and  $\frac{\phi_1}{2}$  to their limiting values.

In the same way, eqs. (16) and (17) become

$$(22) \quad F(k, \phi) = \frac{1}{\cos^2 \frac{\theta}{2} \cos^2 \frac{\theta_0}{2} \dots \cos^2 \frac{\theta_{0n}}{2}} \cdot \frac{\phi_n}{2^n}$$

$$(23) \quad = \sqrt{\frac{\cos \theta_0 \cos \theta_{00} \dots \cos \theta_{0n}}{\cos \theta}} \cdot \frac{\phi_n}{2^n},$$

$n-1$  being the number which makes  $k'_{n-1} = 1$ .

In these last three equations  $k_0, k_{00}$  are determined by eqs. (14<sub>1</sub>);  $\phi_1, \phi_2$ , etc., by eqs. (14<sub>2</sub>)\*;  $\theta, \theta_0$ , etc., by eqs. (14<sub>1</sub>); and  $k', k'_0, k_2$ , etc., for use in eq. (14<sub>2</sub>) by eqs. (14<sub>1</sub>).

---

\* Taking for  $\phi_1 - \phi$ , etc., not always the least angle given by the tables, but that which is nearest to  $\phi$ .

## BISECTED AMPLITUDES.

We have identically

$$u = 2 \cdot \frac{u}{2} = 2 \int \frac{d \operatorname{am} \frac{u}{2}}{\sqrt{1 - k^2 \operatorname{sn}^2 \frac{u}{2}}};$$

$$\frac{u}{2} = 2 \cdot \frac{u}{4} = 2F\left(k, \operatorname{am} \frac{u}{4}\right);$$

etc.

Therefore

$$u = F(k, \operatorname{am} u) = 2^n F\left(k, \operatorname{am} \frac{u}{n}\right)$$

$$= 2^n \cdot \operatorname{am} \frac{u}{n}, \quad (n = \text{limit}),$$

$\operatorname{am} \frac{u}{n}$  being determined by repeated applications of eq. (12) of Chap. II, as follows:

$$\operatorname{sn}^2 \frac{u}{2} = \frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u} = \frac{2 \sin^2 \frac{1}{2} \operatorname{am} u}{1 + \operatorname{dn} u};$$

$$(24) \quad \operatorname{sn} \frac{u}{2} = \frac{\sin \frac{1}{2} \operatorname{am} u}{\sqrt{\frac{1 + \cos \beta}{2}}} = \frac{\sin \frac{\operatorname{am} u}{2}}{\cos \frac{1}{2} \beta};$$

$\beta$  being an angle determined by the equation

$$(25) \quad \cos \beta = \operatorname{dn} u = \sqrt{1 - k^2 \operatorname{sn}^2 u},$$

and  $n$  being the number which makes

$$2^n \operatorname{am} \frac{u}{n} = \text{constant}.$$

$\operatorname{am} \frac{u}{n}$  is found by repeated applications of eq. (24).

Indicating the amplitudes as follows :

$$\text{am } u = \phi,$$

$$\text{am } \frac{u}{2} = \phi_{02},$$

$$\text{am } \frac{u}{4} = \phi_{04},$$

$$\text{am } \frac{u}{8} = \phi_{08},$$

$$\dots$$

$$\text{am } \frac{u}{2^n} = \phi_{02^n}, \dots$$

$$(26) \quad F(k, \phi) = 2^n \phi_{02^n};$$

$n$  being the limiting value.

In eq. (18), when  $k_n$  reaches its limit 1, we have

$$F(k_n, \phi_{0n}) = \int_0^\phi \frac{d\phi_{0n}}{\cos \phi_{0n}} = \log_e \tan (45^\circ + \frac{1}{2}\phi_{0n}),$$

and eqs. (18) and (19) become

$$(27) \quad F(k, \phi) = (1 + k'_0)(1 + k'_{00}) \dots (1 + k'_{0n}) \log_e \tan(45^\circ + \frac{1}{2}\phi_{0n})$$

$$= \frac{1}{\cos^2 \frac{1}{2}\eta \cos^2 \frac{1}{2}\eta_0 \dots \cos^2 \frac{1}{2}\eta_{0n}} \log_e \tan (45^\circ + \frac{1}{2}\phi_{0n})$$

$$(28) \quad = \frac{1}{\cos^2 \frac{1}{2}\eta \cos^2 \frac{1}{2}\eta_0 \dots \cos^2 \frac{1}{2}\eta_{0n}} \cdot \frac{1}{M} \log \tan(45^\circ + \frac{1}{2}\phi_{0n});$$

$n$  being the number which renders  $k_n = 1$ .

Eq. (20) becomes

$$\begin{aligned}
 (29) \quad F(k, \phi) &= \sqrt{\frac{k_1 k_2 \dots k_n^2}{k}} \cdot \log_e \tan (45^\circ + \tfrac{1}{2} \phi_{0n}) \\
 &= \sqrt{\frac{k_1 k_2 \dots k_n^2}{k}} \cdot \frac{1}{M} \log \tan (45^\circ + \tfrac{1}{2} \phi_{0n}) \\
 &= \sqrt{\frac{\cos \eta_0 \cos \eta_{00} \dots \cos^2 \eta_{0n}}{\cos \eta}} \cdot \frac{1}{M} \log \tan (45^\circ + \tfrac{1}{2} \phi_{0n}).
 \end{aligned}$$

In these equations  $k'_0$ ,  $k'_{00}$ , etc., are determined by eqs. (18<sub>3</sub>);  $\eta$ ,  $\eta_0$ , etc., by eqs. (18<sub>3</sub>);  $\phi_0$ ,  $\phi_{00}$ , etc., by eqs. (18<sub>2</sub>);  $k_1$ ,  $k_2$ , etc., by eqs. (18<sub>1</sub>).

Substituting in eq. (27) from eq. (20)\*, we have

$$\begin{aligned}
 F(k, \phi) &= \frac{2K'}{\pi} \log_e \tan (45^\circ + \tfrac{1}{2} \phi_{0n}) \\
 (30) \quad &= \frac{2K'}{\pi M} \log \tan (45^\circ + \tfrac{1}{2} \phi_{0n}).
 \end{aligned}$$

## CHAPTER V.

### COMPLETE FUNCTIONS.

INDICATE by  $K$  the complete integral

$$(1) \quad K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

and by  $K_0$  the complete integral

$$(2) \quad K_0 = \int_0^{\frac{\pi}{2}} \frac{d\phi_1}{\sqrt{1 - k_0'^2 \sin^2 \phi_1}};$$

and in a similar manner  $K_{00}$ ,  $K_{03}$ , etc.

From eq. (12), Chap. IV, we have

$$\begin{aligned} \tan(\phi_1 - \phi) &= k' \tan \phi \\ &= \frac{\tan \phi_1 - \tan \phi}{1 + \tan \phi_1 \tan \phi}, \end{aligned}$$

whence

$$\begin{aligned} \tan \phi_1 &= \frac{(1 + k') \tan \phi}{1 - k' \tan^2 \phi} \\ &= \frac{1 + k'}{\frac{1}{\tan \phi} - k' \tan \phi}. \end{aligned}$$

From this equation we see that when  $\phi = \frac{\pi}{2}$ ,  $\phi_1 = \pi$ . This same result might also have been deduced from Fig. 1, Chap. IV, or from the equation

$$(3) \quad \phi_1 = 2\phi - k_0 \sin 2\phi + \frac{1}{2}k_0^2 \sin 4\phi - \text{etc.},$$

this last being the well-known trigonometrical formula

$$\tan x = n \tan y,$$

$$x = y - \frac{1-n}{1+n} \sin 2y + \frac{1}{2} \left( \frac{1-n}{1+n} \right)^2 \sin 4y - \frac{1}{3} \left( \frac{1-n}{1+n} \right)^3 \sin 6y + \text{etc.}$$

Since  $\int_0^{\frac{\pi}{2}} \frac{d\phi_1}{\Delta(k_0\phi_1)} = K_0$ , we must have

$$\int_0^{\pi} \frac{d\phi_1}{\Delta(k_0\phi_1)} = 2K_0.$$

These values substituted in eq. (13), Chap. IV, give successively

$$(4) \quad \begin{aligned} K &= (1 + k_0)K_0, \\ K_0 &= (1 + k_{00})K_{00}, \\ &\dots \dots \dots \\ K_{0(n-1)} &= (1 + k_{0n})K_{0n}; \end{aligned}$$

whence

$$(5) \quad K = (1 + k_0)(1 + k_{00}) \dots (1 + k_{0n})K_{0n}.$$

Since the limit of  $k_{0n}$  is 0,  $K_{0n}$  becomes

$$K_{0n} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2},$$

and we have

$$(6) \quad K = \frac{\pi}{2} (1 + k_0)(1 + k_{00}) \dots$$

$$(7) \quad = \frac{\frac{1}{2}\pi}{\cos^2 \frac{1}{2}\theta \cos^2 \frac{1}{2}\theta_0 \dots \cos^2 \frac{1}{2}\theta_{0n}};$$

$k_1, k_0$ , etc., and  $\theta_1, \theta_0$ , etc., being found by eqs. (14<sub>1</sub>) of Chap. IV.

From the formulæ in these two chapters we can compute the values of  $u$  for all values of  $\phi$  and  $k$  and arrange them in tables. These are Legendre's Tables of Elliptic Integrals.

## CHAPTER VI.

EVALUATION FOR  $\phi$ 

TO FIND  $\phi$ ,  $u$  AND  $k$  BEING GIVEN.

From eqs. (21) and (23), Chap. IV, we have ( $n$  having the value which makes  $\cos \theta_{on} = 1$ )

$$(I) \quad \phi_n = \frac{2^n u}{(1+k_0)(1+k_{00}) \dots (1+k_{0n})} = \frac{2^n u \sqrt{\cos \theta}}{\sqrt{\cos \theta_0} \dots \cos^2 \theta_{0n}},$$

from which  $\phi_n$  can be calculated,  $k_0$ ,  $k_{00}$ , etc., being found by means of equations (I<sub>4</sub>), Chap. IV.

Then, having  $\phi_n, k_0, k_{00}$ , etc., we can find  $\phi$  by means of the following equations:

[illegible]

whence we can get the angle  $\phi$ .

When  $k > 1\frac{1}{2}$  the following formulæ will generally be found to work more rapidly:

From eq. (29), Chap. IV, we have

$$(2) \quad \log \tan (45^{\circ} + \frac{1}{2} \phi_{on}) = \frac{uM}{\sqrt{\frac{k_1 k_2 \dots k_n^2}{k}}},$$

from which we can get  $\phi_{0n}$ ;  $k_1$ ,  $k_2$ , etc., being calculated from eqs. (18<sub>1</sub>), Chap. IV, and  $\phi$  being calculated from the following equations:

$$\begin{aligned}\tan(\phi_{0(n-1)} - \phi_{0n}) &= k_n \tan \phi_{0n}, \\ &\dots \dots \dots \\ \tan(\phi_0 - \phi_{00}) &= k_2 \tan \phi_{02}, \\ \tan(\phi - \phi_0) &= k \tan \phi_0;\end{aligned}$$

whence we get  $\phi$ .

This gives a method of solving the equation

$$F\psi = nF\phi,$$

where  $n$  and  $\phi$  and the moduli are known, and  $\psi$  is the required quantity.  $n$  and  $\phi$  give  $F\psi$ , and then  $\psi$  can be determined by the foregoing methods.

When  $k = 1$  nearly, equation (2) takes a special form,—

1°. When  $\tan \phi$  is very much less than  $\frac{1}{k'}$ . In this case

$$\begin{aligned}F(k, \phi) &= \int \frac{d\phi}{\sqrt{\cos^2 \phi + k'^2 \sin^2 \phi}} = \int \frac{d\phi}{\sqrt{(1 + k'^2 \tan^2 \phi) \cos^2 \phi}} \\ &= \int \frac{d\phi}{\cos \phi} = \log \tan (45^\circ + \tfrac{1}{2}\phi);\end{aligned}$$

whence we can find  $\phi$ .

2°. When  $\tan \phi$  and  $\frac{1}{k'}$  approach somewhat the same value, and  $k' \tan \phi$  cannot be neglected,  $F(k, \phi)$  must be transposed into another where  $k'$  shall be much smaller, so that  $k' \tan \phi$  can be neglected.

These methods for finding  $\phi$  apply only when  $\phi < \frac{\pi}{2}$ , that is,  $u < K$ . In the opposite case ( $u > K$ ) put

$$u = 2nK \pm v,$$

the upper or the lower sign being taken according as  $K$  is continued in  $u$  an even or an odd number of times. In either case  $v < K$ , and we can find  $v$  by the preceding methods.

Having found  $v$ , we have from eq. (5), Chap. III,

$$\begin{aligned} \text{am } u &= \text{am } (2nK \pm v) \\ &= n\pi \pm \text{am } v. \end{aligned}$$

## CHAPTER VII.

### DEVELOPMENT OF ELLIPTIC FUNCTIONS INTO FACTORS.

FROM eq. (12), Chap. IV, we readily get

$$\sin (2\phi_0 - \phi) = k \sin \phi;$$

$$\begin{aligned} \sin \phi &= \frac{\sin 2\phi_0}{\sqrt{1+k^2+2k \cos 2\phi_0}} \\ &= \frac{\sin 2\phi_0}{\sqrt{(1+k^0) - 4k \sin^2 \phi_0}} \\ &= \frac{1+k_0'}{2} \cdot \frac{\sin 2\phi_0}{\sqrt{1-k_1^2 \sin^2 \phi_0}} \end{aligned}$$

(since  $\frac{4k}{1+k} = k_1$  and  $1+k = \frac{2}{1+k_0'}$ , eqs. (6) and (10), Chap. IV);  
and thence

$$(1) \quad \sin \phi = \frac{(1+k_0') \sin \phi_0 \cos \phi_0}{\Delta(\phi_0, k_1)}.$$

From eq. (13), Chap. IV, we have

$$\int_0^{\phi_0} \frac{d\phi_0}{\Delta(\phi_0, k_1)} = \frac{1+k}{2} \int_0^{\phi} \frac{d\phi}{\Delta(\phi, k)};$$

and from eq. (4), Chap. V, passing up the scale of moduli one step,

$$1+k = \frac{K_1}{K},$$

whence

$$F(\phi_0, k_1) = \frac{K_1}{2K} F(\phi, k).$$

Put

$$F(\phi_0, k_1) = u_1 \quad \text{and} \quad F(\phi, k) = u,$$

whence

$$u_1 = \frac{K_1}{2K} u.$$

Furthermore,

$$\phi = \text{am } (u, k);$$

$$\phi_1 = \text{am } (u_1, k_1) = \text{am } \left( \frac{K_1}{2K} u, k_1 \right).$$

Substituting these values in eq. (1), we have

$$\text{sn } (u, k) = (1 + k_0') \frac{\text{sn } \left( \frac{K_1}{2K} u, k_1 \right) \text{cn } \left( \frac{K_1}{2K} u, k_1 \right)}{\text{dn } \left( \frac{K_1}{2K} u, k_1 \right)}.$$

But from eq. (11), Chap. III, we have

$$\frac{\text{cn } (v, k_1)}{\text{dn } (v, k_1)} = \text{sn } (v + K_1, k_1),$$

or

$$\begin{aligned} \frac{\text{cn } \left( \frac{K_1}{2K} u, k_1 \right)}{\text{dn } \left( \frac{K_1}{2K} u, k_1 \right)} &= \text{sn } \left( \frac{K_1 u}{2K} + K_1, k_1 \right) \\ &= \text{sn } \left( \frac{K_1}{2K} (u + K_1), k_1 \right); \end{aligned}$$

whence

$$(2) \quad \text{sn}(u, k) = (1 + k'_0) \text{sn} \frac{K_1 u}{2K} \text{sn} \left[ \frac{K_1}{2K} (u + 2K) \right]^\dagger$$

(Mod. =  $k_1$ .)

From this equation, evidently, we have generally

$$(2)^* \quad \text{sn}(\nu, k_n) = (1 + k'_0{}_{(n+1)}) \text{sn} \frac{K_{n+1} \nu}{2K_n} \text{sn} \left[ \frac{K_{n+1}}{2K_n} (\nu + 2K_n) \right].$$

(Mod. =  $k_{n+1}$ .)

Applying this general formula to the two factors of eq. (2), we have

$$\text{sn} \left( \frac{K_1 u}{2K}, k_1 \right) = (1 + k'_{00}) \text{sn} \frac{K_2}{2K_1} \cdot \frac{K_1 u}{2K} \cdot \text{sn} \left[ \frac{K_2}{2K_1} \left( \frac{K_1 u}{2K} + 2K_1 \right) \right]$$

(Mod.  $k_2$ )

$$= (1 + k'_{00}) \text{sn} \frac{K_2 u}{2^2 K} \text{sn} \frac{K_2}{2^2 K} (u + 4K); \quad (\text{Mod. } k_2;)$$

$$(3) \quad \text{sn} \left[ \frac{K_1}{2K} (u + 2K), k_1 \right] = (1 + k'_{00}) \text{sn} \frac{K_2}{2^2 K} (u + 2K)$$

$$\cdot \text{sn} \frac{K_2}{2K_1} \left[ \frac{K_1}{2K} (u + 2K) + 2K_1 \right]. \quad (\text{Mod. } k_2.)$$

The last argument in this equation is equal to

$$\frac{K_2}{2^2 K} (u + 6K);$$

and since, eq. (7), Chap. III,

$$\text{sn}(u, k_2) = \text{sn}(2K_2 - u, k_2),$$

† The analogous formula in Trigonometry is

$$\sin \phi = \frac{1}{2} \sin \frac{1}{2} \phi \sin \frac{1}{2} (\phi + \pi).$$

we can put in place of this,

$$2K_2 - \frac{K_2}{2^2 K} (u + 6K) = \frac{K_2}{2^2 K} (2K - u);$$

whence eq. (3) becomes

$$\begin{aligned} \operatorname{sn} \left[ \frac{K_1}{2K} (u + 2K), k_1 \right] &= (1 + k'_{00}) \operatorname{sn} \frac{K_2}{2^2 K} (2K + u) \\ &\cdot \operatorname{sn} \frac{K_2}{2^2 K} (2K - u). \end{aligned} \quad (\text{Mod. } k_2.)$$

Substituting these values in eq. (2), we have

$$\begin{aligned} (4) \quad \operatorname{sn} (u, k) &= (1 + k'_0)(1 + k'_{00})^2 \operatorname{sn} \frac{K_2 u}{2^2 K} \\ &\cdot \operatorname{sn} \frac{K_2}{2^2 K} (2K \pm u) \operatorname{sn} \frac{K_2}{2^2 K} (4K + u), \end{aligned} \quad (\text{Mod. } k_2.)$$

in which the double sign indicates two separate factors which are to be multiplied together.

By the application of the general equation (2)\* we find that the arguments in the second member of eq. (4) will each give rise to two new arguments, as follows:

$$\frac{K_2 u}{2^2 K} \text{ gives } \frac{K_3 u}{2^2 K},$$

and

$$\frac{K_3}{2K_2} \left( \frac{K_2 u}{2^2 K} + 2K_2 \right) = \frac{K_3}{2^2 K} (u + 8K);$$

$$\frac{K_2}{2^2 K} (2K \pm u) \text{ gives } \frac{K_3}{2^2 K} (2K \pm u),$$

and

$$\frac{K_3}{2K_2} \left[ \frac{K_2}{2^2 K} (2K \pm u) + 2K_2 \right] = \frac{K_3}{2^3 K} (10K \pm u), \quad . \quad . \quad (a)$$

$$\frac{K_3}{2^2 K} (4K + u) \quad \text{gives} \quad \frac{K_3}{2^3 K} (4K + u),$$

and

$$\frac{K_3}{2K_2} \left[ \frac{K_2}{2^2 K} (4K + u) + 2K_2 \right] = \frac{K_3}{2^3 K} (12K + u). \quad . \quad . \quad (b)$$

Subtracting (a) and (b) from  $2K_3$ , by which the sine of the amplitudes will not be changed [eq. (7), Chap. III], and since our new modulus is  $k_3$ , we have for the expressions (a) and (b),

$$\frac{K_3}{2^3 K} (6K \mp u); \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

$$\frac{K_3}{2^3 K} (4K - u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

Substituting these values in eq. (4), and remembering the factor  $(1 + k'_{03})$  introduced by each application of eq. (2)\*, we have

$$\begin{aligned} \operatorname{sn}(u, k) &= (1 + k'_0)(1 + k'_{00})^2(1 + k'_{03})^4 \operatorname{sn} \frac{K_3 u}{2^3 K}; \\ &\cdot \operatorname{sn} \frac{K_3}{2^3 K} (2K \pm u) \operatorname{sn} \frac{K_3}{2^3 K} (4K \pm u); \\ &\cdot \operatorname{sn} \frac{K_3}{2^3 K} (6K \pm u) \operatorname{sn} \frac{K_3}{2^3 K} (8K + u). \quad (\text{Mod. } k_3.) \end{aligned}$$

From this the law governing the arguments is clear, and we can write for the general equation

$$\begin{aligned}
 (5) \quad \operatorname{sn}(u, k) &= (1 + k'_0)(1 + k'_{00})^2(1 + k'_{03})^4 \dots (1 + k'_{0n})^{2^{n-1}} \\
 &\quad \cdot \operatorname{sn} \frac{K_n u}{2^n K} \operatorname{sn} \frac{K_n}{2^n K} (2K \pm u) \\
 &\quad \cdot \operatorname{sn} \frac{K_n}{2^n K} (4K \pm u) \operatorname{sn} \frac{K_n}{2^n K} (6K \pm u) \\
 &\quad \cdot \dots \operatorname{sn} \frac{K_n}{2^n K} \left[ (2^n - 2)K \pm u \right] \\
 &\quad \cdot \operatorname{sn} \frac{K_n}{2^n K} (2^n K \pm u). \quad (\text{Mod. } k_n)
 \end{aligned}$$

Indicate the continued product of the binomial factors by  $A'$ , and we have

$$A' = (1 + k'_0)(1 + k'_{00})^2(1 + k'_{03})^4(1 + k'_{04})^8 \dots$$

Since the limit of  $k'_0$ ,  $k'_{00}$ , etc., is zero, it is evident that these factors converge toward the value unity. It can be shown that the functional factors also converge toward the value unity. Thus the argument of the last factor can be written

$$K_n + \frac{K_n u}{2^n K}.$$

From eq. (11), Chap. III, we get then

$$\operatorname{sn} \left( K_n + \frac{K_n u}{2^n K} \right) = \frac{\operatorname{cn} \frac{K_n u}{2^n K}}{\operatorname{dn} \frac{K_n u}{2^n K}}. \quad (\text{Mod. } k_n)$$

But since  $k_n$  at its limit is equal to unity,  $\operatorname{cn} = \operatorname{dn}$ ; whence the last factor of eq. (5) is unity.

From eq. (21), Chap. IV, we have

$$\lim_{n \rightarrow \infty} \frac{K_n}{2^n K} = \frac{2\pi}{2K'}.$$

Therefore for  $n = \infty$ , eq. (5) becomes

$$\begin{aligned} \operatorname{sn}(u, k) &= A' \operatorname{sn} \frac{\pi u}{2K'} \operatorname{sn} \frac{\pi}{2K'} (2K \pm u) \\ &\quad \cdot \operatorname{sn} \frac{\pi}{2K'} (4K \pm u) \operatorname{sn} \frac{\pi}{2K'} (6K \pm u), \quad \dots \dots \dots \\ &\quad (\text{Mod. } 1,) \end{aligned}$$

or

$$(6) \quad \operatorname{sn}(u, k) = A' \operatorname{sn} \frac{\pi u}{2K'} \left[ \prod_i^{\infty} \right] \operatorname{sn} \frac{\pi}{2K'} (2hK \pm u),$$

(Mod. 1,)

where the sign  $[\prod]$  indicates the continued product in the same manner as  $\Sigma$  indicates the continued sum.

When  $k = 1$ ,  $\int_0^\phi F(\phi, k)$  becomes

$$\nu = \int_0^\phi \frac{d\phi}{\cos \phi} = \frac{1}{2} \log \frac{1 + \sin \phi}{1 - \sin \phi};$$

whence

$$e^{2\nu} = \frac{1 + \sin \phi}{1 - \sin \phi},$$

and

$$\sin \phi = \frac{e^{2\nu} - 1}{e^{2\nu} + 1} = \frac{e^\nu - e^{-\nu}}{e^\nu + e^{-\nu}} = \operatorname{sn}(\nu, 1).$$

Hence in equation (6)

$$\sin \frac{\pi u}{2K'} = \frac{e^{\frac{\pi u}{2K'}} - e^{-\frac{\pi u}{2K'}}}{e^{\frac{\pi u}{2K'}} + e^{-\frac{\pi u}{2K'}}};$$

$$\operatorname{sn} \frac{\pi}{2K'} (2hK \pm u) = \frac{e^{\frac{h\pi K}{K'}} e^{\pm \frac{\pi u}{2K'}} - e^{-\frac{h\pi K}{K'}} e^{\mp \frac{\pi u}{2K'}}}{e^{\frac{h\pi K}{K'}} e^{\pm \frac{\pi u}{2K'}} + e^{-\frac{h\pi K}{K'}} e^{\mp \frac{\pi u}{2K'}}}.$$

Put

$$(6)^* \quad q = e^{-\frac{\pi K'}{K}}, \quad q' = e^{-\frac{\pi K}{K'}},$$

and the last expression becomes

$$\begin{aligned} \operatorname{sn} \frac{\pi}{2K'} (2hK \pm u) &= \frac{q'^{-h} e^{\pm \frac{\pi u}{2K'}} - q'^h e^{\mp \frac{\pi u}{2K'}}}{q'^{-h} e^{\pm \frac{\pi u}{2K'}} + q'^h e^{\mp \frac{\pi u}{2K'}}}; \\ \operatorname{sn} \frac{\pi}{2K'} (2hK + u) \operatorname{sn} \frac{\pi}{2K'} (2hK - u) &= \frac{q'^{-h} e^{\frac{\pi u}{2K'}} - q'^h e^{-\frac{\pi u}{2K'}}}{q'^{-h} e^{\frac{\pi u}{2K'}} + q'^h e^{-\frac{\pi u}{2K'}}} \cdot \frac{q'^{-h} e^{-\frac{\pi u}{2K'}} - q'^h e^{\frac{\pi u}{2K'}}}{q'^{-h} e^{-\frac{\pi u}{2K'}} + q'^h e^{\frac{\pi u}{2K'}}} \\ &= \frac{q'^{-2h} + q'^{2h} - \left( e^{\frac{\pi u}{K'}} + e^{-\frac{\pi u}{K'}} \right)}{q'^{-2h} + q'^{2h} + \left( e^{\frac{\pi u}{K'}} + e^{-\frac{\pi u}{K'}} \right)}. \end{aligned}$$

From plane trigonometry we have the equations

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = -i \tan ix, \quad e^x + e^{-x} = 2 \cos ix;$$

where  $i = \sqrt{-1}$ : which gives

$$\operatorname{sn} \frac{\pi u}{2K'} = -i \tan \frac{\pi i u}{2K'}; \quad (\text{Mod. } 1;)$$

$$\operatorname{sn} \frac{\pi}{2K'} (2hK + u) \operatorname{sn} \frac{\pi}{2K'} (2hK - u)$$

$$= \frac{q'^{-2h} + q'^{2h} - 2 \cos \frac{\pi i u}{K'}}{q'^{-2h} + q'^{2h} + 2 \cos \frac{\pi i u}{K'}}$$

$$= \frac{1 - 2q'^{2h} \cos \frac{\pi i u}{K'} + q'^{4h}}{1 + 2q'^{2h} \cos \frac{\pi i u}{K'} + q'^{4h}}.$$

From eq. (10), Chap. III, we have

$$\operatorname{sn} (u, k) = -i \operatorname{tn} (iu, k').$$

Substituting these values in eq. (6), we have

$$\operatorname{tn} (iu, k') = A' \tan \frac{\pi i u}{2K'} \left[ \Pi \right] \frac{1 - 2q'^{2h} \cos \frac{\pi i u}{K'} + q'^{4h}}{1 + 2q'^{2h} \cos \frac{\pi i u}{K'} + q'^{4h}}.$$

Now in place of the series of moduli  $k'$ ,  $k'_0$  and the corresponding complete integral  $K'$ , we are at liberty to substitute the parallel series of moduli  $k$ ,  $k_0$  and the corresponding complete integral  $K$ ; calling the new integral  $u$ , we have

$$(8) \quad \operatorname{tn} (u, k) = A \tan \frac{\pi u}{2K} \Pi \frac{1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 + 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}$$

$$\begin{aligned}
&= A \tan \frac{\pi u}{2K} \frac{1 - 2q^2 \cos \frac{\pi u}{K} + q^4}{1 + 2q^2 \cos \frac{\pi u}{K} + q^4} \\
&\quad \cdot \frac{1 - 2q^4 \cos \frac{\pi u}{K} + q^8}{1 + 2q^4 \cos \frac{\pi u}{K} + q^8} \\
&\quad \cdot \frac{1 - 2q^6 \cos \frac{\pi u}{K} + q^{12}}{1 + 2q^6 \cos \frac{\pi u}{K} + q^{12}} \dots,
\end{aligned}$$

where

$$(9) \quad A = (1 + k_0)(1 + k_{00})^2(1 + k_{03})^4(1 + k_{04})^8 \dots$$

Now in equation (6) put  $u + K$  for  $u$ , and we have, since [eq. (11), Chap. III]  $\operatorname{sn}(u + K) = \frac{\operatorname{cn} u}{\operatorname{dn} u}$ ,

$$\begin{aligned}
\frac{\operatorname{cn} u}{\operatorname{dn} u} &= A' \operatorname{sn} \frac{\pi(u + K)}{2K'} \left[ II \right] \operatorname{sn} \frac{\pi}{2K'} [(2h + 1)K + u] \\
&\quad \cdot \operatorname{sn} \frac{\pi}{2K'} [(2h - 1)K - u]. \quad (\text{Mod. } 1.)
\end{aligned}$$

Now from  $2h - 1$  and  $2h + 1$  we have the following series of numbers respectively :

$$2h - 1 : \quad 1, \quad 3, \quad 5, \quad 7, \quad 9, \quad \text{etc.}$$

$$2h + 1 : \quad 3, \quad 5, \quad 7, \quad 9, \quad \text{etc.}$$

It will be observed that the factor outside of the sign  $[II]$ , viz.,  $\sin \operatorname{am} \frac{\pi(u + K)}{2K'}$ , would, if placed under the sign  $[II]$ , supply

the missing first term of the second series. Hence, placing this factor within the sign, we have

$$(10) \quad \frac{\text{cn } u}{\text{dn } u} = A' [II] \text{sn } \frac{\pi}{2K'} [(2h-1)K + u] \\ \cdot \text{sn } \frac{\pi}{2K'} [(2h-1)K - u]. \quad (\text{Mod. } 1.)$$

Comparing this with equation (7), we see that the factors herein differ from those in equation (7) only in having  $2h-1$  in place of  $2h$ ; hence we have

$$\text{sn } \frac{\pi}{2K'} [(2h-1)K + u] \text{sn } \frac{\pi}{2K'} [(2h-1)K - u] \\ = \frac{1 - 2q'^{2h-1} \cos \frac{\pi iu}{K'} + q'^{4h-2}}{1 + 2q'^{2h-1} \cos \frac{\pi iu}{K'} + q'^{4h-2}}. \quad (\text{Mod. } 1.)$$

From eqs. (10), Chap. III, we have

$$\frac{\text{cn } (u, k)}{\text{dn } (u, k)} = \frac{1}{\text{dn } (iu, k')};$$

whence eq. (10) becomes

$$(11) \quad \frac{1}{\text{dn } (iu, k')} = A' [II] \frac{1 - 2q'^{2h-1} \cos \frac{\pi iu}{K'} + q'^{4h-2}}{1 + 2q'^{2h-1} \cos \frac{\pi iu}{K'} + q'^{4h-2}};$$

and when in place of  $iu, k', K', q', A'$ , we substitute  $u, k, K, q$  and  $A$ , and invert the equation, we have

$$(12) \quad \text{dn } (u, k) = \frac{1}{A} [II] \frac{1 + 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}.$$

Bearing in mind the remarkable property (Chap. III, p. 29) that the functions  $\text{sn } u$  and  $\text{dn } u$  approach infinity for the same value of  $u$ , we see that both these functions, except as to the factor independent of  $u$ , must have the same denominator. Furthermore, since  $\text{sn } u$  and  $\text{tn } u$  disappear for the same value of  $u$ , they must, except for the independent factor, have the same numerator. Hence, indicating by  $B$  a new quantity, dependent upon  $k$  but independent of  $u$ , we have

$$(13) \quad \text{sn } (u, k) = B \sin \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}};$$

and since

$$\text{cn } u = \frac{\text{sn } u}{\text{tn } u},$$

we also have, from eqs. (8) and (13),

$$(14) \quad \text{cn } (u, k) = \frac{B}{A} \cos \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 + 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}.$$

Collecting these results, we have the following equations :

$$(15) \quad \text{sn } (u, k) = B \sin \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}},$$

$$(16) \quad \text{cn } (u, k) = \frac{B}{A} \cos \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 + 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}},$$

$$(17) \quad \text{dn } (u, k) = \frac{1}{A} \left[ \Pi \right] \frac{1 + 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}.$$

To ascertain the values of  $A$  and  $B$ , we proceed as follows:

In eq. (17) we make  $u = 0$ , whence, by eq. (13), Chap. II, we have

$$1 = \frac{1}{A} \left[ \Pi \right] \left( \frac{1 + q^{2h-1}}{1 - q^{2h-1}} \right)^2;$$

whence

$$(18) \quad \frac{1}{A} = \left[ \Pi \right] \left( \frac{1 - q^{2h-1}}{1 + q^{2h-1}} \right)^2.$$

In equation (17), making  $u = K$ , we get, by equation (1), Chap. III,

$$k' = \frac{1}{A} \left[ \Pi \right] \left( \frac{1 - q^{2h-1}}{1 + q^{2h-1}} \right)^2 = \frac{1}{A^2};$$

$$(19) \quad \therefore \frac{1}{A} = \sqrt{k'}.$$

whence

$$\frac{B}{A} = B \sqrt{k'}.$$

To calculate  $B$ , put  $e^{\frac{i\pi u}{2K}} = v$ ; if we change  $\frac{\pi u}{2K}$  into

$\frac{\pi u}{2K} + \frac{i\pi K'}{2K}$ ,  $v$  will change into  $v\sqrt{q}$ , and  $\operatorname{sn} u$  will become, by eq. (14), Chap. III,

$$\operatorname{sn}(u + iK') = \frac{1}{k \operatorname{sn} u}.$$

Now, replacing  $\sin \frac{\pi u}{2K}$  and  $\cos \frac{\pi u}{K}$  by their exponential values, and observing that

$$1 - 2q^n \cos \frac{\pi u}{K} + q^{2n} = (1 - q^n v^2)(1 - q^n v^{-2}),$$

we have

$$\operatorname{sn} u = \frac{B}{2} \cdot \frac{v - v^{-1}}{\sqrt{1 - 1}} \cdot \frac{[II](1 - q^{2h} v^2)(1 - q^{2h} v^{-2})}{[II](1 - q^{2h-1} v^2)(1 - q^{2h-1} v^{-2})}.$$

Changing  $u$  into  $u + iK'$ , and consequently  $v$  into  $v\sqrt{q}$ , we have

$$\frac{1}{k \operatorname{sn} u} = \frac{B}{2} \cdot \frac{v\sqrt{q} - v^{-1}\sqrt{q^{-1}}}{\sqrt{1 - 1}} \cdot \frac{[II](1 - q^{2h+1} v^2)(1 - q^{2h-1} v^{-2})}{[II](1 - q^{2h} v^2)(1 - q^{2h-2} v^2)}.$$

Multiplying these equations together, member by member, and observing that

$$v\sqrt{q} - v^{-1}\sqrt{q^{-1}} = \frac{1 - qv^2}{-v\sqrt{q}},$$

$$v - v^{-1} = v(1 - v^{-2}),$$

we get

$$\frac{1}{k} = \frac{B^2}{4} \cdot \frac{1 - qv^2}{v\sqrt{q}} \cdot v(1 - v^{-2}) \cdot \frac{[II](1 - q^{2h+1} v^2)(1 - q^{2h} v^{-2})}{[II](1 - q^{2h-1} v^2)(1 - q^{2h-2} v^2)}$$

$$\begin{aligned}
&= \frac{B^2}{4\sqrt{q}} (1 - q\nu^2) (1 - \nu^{-2}) \frac{(1 - q^3\nu^2)(1 - q^5\nu^2) \dots}{(1 - q\nu^2)(1 - q^3\nu^2) \dots} \\
&\quad \cdot \frac{(1 - q^2\nu^{-2})(1 - q^4\nu^{-2}) \dots}{(1 - \nu^{-2})(1 - q^2\nu^{-2}) \dots} \\
&= \frac{B^2}{4} \cdot \frac{1}{\sqrt{q}}.
\end{aligned}$$

$$\therefore B = \frac{2\sqrt[4]{q}}{\sqrt{k}};$$

whence

$$\frac{B}{A} = 2\sqrt[4]{q}\sqrt{\frac{k'}{k}}.$$

Substituting these values in eqs. (15), (16), and (17), we have

$$(20) \quad \operatorname{sn}(u, k) = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}};$$

$$(21) \quad \operatorname{cn}(u, k) = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}} \cos \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 + 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}};$$

$$(22) \quad \operatorname{dn}(u, k) = \sqrt{k'} \left[ \Pi \right] \frac{1 + 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}.$$

## CHAPTER VIII.

### THE $\Theta$ FUNCTION.

WE will indicate the denominator in eq. (20), Chap. VII, by  $\phi(u)$ , thus:

$$(1) \quad \phi(u) = [\Pi](1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}).$$

We will now develop this into a series consisting of the cosines of the multiples of  $\frac{\pi u}{K}$ . Put  $\frac{\pi u}{2K} = x$ , whence

$$2 \cos \frac{\pi u}{K} = (e^{2ix} + e^{-2ix});$$

but

$$1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2} = (1 - q^{2h-1} e^{2ix})(1 - q^{2h-1} e^{-2ix}),$$

and therefore

$$(2) \quad \begin{aligned} \phi(u) &= (1 - qe^{2ix})(1 - q^3e^{2ix})(1 - q^5e^{2ix}) \dots \\ &\quad (1 - qe^{-2ix})(1 - q^3e^{-2ix})(1 - q^5e^{-2ix}) \dots \end{aligned}$$

Putting now  $u + 2iK'$  instead of  $u$ , we have

$$x_1 = \frac{\pi(u + 2iK')}{2K} = x + \frac{\pi iK'}{K},$$

$$2ix_1 = 2ix - \frac{2\pi K'}{K};$$

and

$$e^{2ix_1} = q^2 e^{2ix},$$

$$e^{-2ix_1} = \frac{1}{q^2} e^{-2ix}.$$

From these we have

$$\begin{aligned} \phi(u + 2iK') &= -\frac{1}{q} e^{-2ix} (1 - qe^{2ix})(1 - q^3 e^{2ix}) \dots \\ &\quad (1 - qe^{-2ix})(1 - q^3 e^{-2ix}) \dots; \end{aligned}$$

whence

$$\phi(u + 2iK') = -\frac{1}{q} e^{-2ix} \phi(u),$$

or

$$(3) \quad \phi(u + 2iK') = -q^{-1} e^{-\frac{\pi i u}{K}} \phi(u).$$

Now put

$$(4) \quad \phi(u) = A + B \cos \frac{\pi u}{K} + C \cos \frac{2\pi u}{K} + D \cos \frac{3\pi u}{K} + \text{etc.}$$

Since

$$\cos \frac{\pi u}{K} = \frac{1}{2}(e^{2ix} + e^{-2ix}),$$

this becomes

$$\begin{aligned} (5) \quad \phi(u) &= A + \frac{1}{2}B e^{2ix} + \frac{1}{2}C e^{4ix} + \frac{1}{2}D e^{6ix} + \dots \\ &\quad + \frac{1}{2}B e^{-2ix} + \frac{1}{2}C e^{-4ix} + \frac{1}{2}D e^{-6ix} + \dots; \end{aligned}$$

whence

$$\begin{aligned} (6) \quad -\frac{1}{q} e^{-2ix} \phi(u) &= -\frac{A}{q} e^{-2ix} - \frac{B}{2q} - \frac{C}{2q} e^{2ix} - \frac{D}{2q} e^{4ix} - \dots \\ &\quad - \frac{B}{2q} e^{-4ix} - \frac{C}{2q} e^{-6ix} - \frac{D}{2q} e^{-8ix} - \dots \end{aligned}$$

Now in equation (5) put  $u + 2iK'$  in place of  $u$ , remembering that  $e^{2ix}$  and  $e^{-2ix}$  are thereby changed respectively into  $q^2 e^{2ix}$  and  $q^{-2} e^{-2ix}$ , and we have

$$(7) \quad \phi(u + 2iK') = A + \frac{Bq^2}{2} e^{2ix} + \frac{Cq^4}{2} e^{4ix} + \frac{Dq^6}{2} e^{6ix} + \dots \\ + \frac{B}{2q^2} e^{-2ix} + \frac{C}{2q^4} e^{-4ix} + \dots$$

Since equations (6) and (7) are equal, we have

$$\begin{aligned} -\frac{B}{2q} &= A, & B &= -2qA; \\ -\frac{C}{2q} &= \frac{Bq^2}{2}, & C &= +2q^4A; \\ -\frac{D}{2q} &= \frac{Cq^4}{2}, & D &= -2q^9A; \\ &\dots & &\dots \end{aligned}$$

whence

$$(8) \quad \left\{ \begin{aligned} &[II](1 - 2q^{2k-1} \cos \frac{\pi u}{K} + q^{4k-2}) \\ &= A(1 - 2q \cos \frac{\pi u}{K} + 2q^4 \cos \frac{2\pi u}{K} - 2q^9 \cos \frac{3\pi u}{K} \\ &\quad + 2q^{16} \cos \frac{4\pi u}{K} - \dots). \end{aligned} \right.$$

The series in the second member has been designated by Jacobi and subsequent writers by  $\Theta(u)$ , thus:

$$(9) \quad \Theta(u) = 1 - 2q \cos \frac{\pi u}{K} + 2q^4 \cos \frac{2\pi u}{K} - \dots$$

## CHAPTER IX.

### THE $\Theta$ AND $H$ FUNCTIONS.

IN equation (20), Chap. VII, viz.,

$$\operatorname{sn}(u, k) = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \left[ II \right] \frac{1 - 2q^{2h} \cos \frac{u\pi}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}},$$

the numerator and the denominator have been considered separately by Jacobi, who gave them a special notation and developed from them a theory second only in importance to the elliptic functions themselves.

Put [see equation (8), Chap. VIII]

$$(1) \quad \Theta(u) = \frac{1}{A} \left[ II \right] (1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}).$$

$$(2) \quad H(u) = 2 \frac{1}{A} \sqrt[4]{q} \sin \frac{\pi u}{2K} \left[ II \right] (1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h});$$

$A$  being a constant whose value is to be determined later. From these we have

$$(3) \quad \operatorname{sn}(u, k) = \frac{1}{\sqrt{k}} \cdot \frac{H(u)}{\Theta(u)}.$$

The functions  $\text{sn } u$  and  $\text{cn } u$  can also be expressed in terms of the new functions; thus we have

$$(4) \quad \text{cn } (u, k) = \sqrt{\frac{k'}{k}} \cdot 2 \sqrt{q} \cos \frac{\pi u}{2K} \left[ \Pi \right] \frac{1 + 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}},$$

or, since  $\sin x = \cos \left( x + \frac{\pi}{2} \right)$  and  $\cos x = -\cos \left( x + \frac{\pi}{2} \right)$ ,

and putting  $u = \frac{2Kx}{\pi}$ ,

$$\begin{aligned} \text{cn } \left( \frac{2Kx}{\pi}, k \right) &= \sqrt{\frac{k'}{k}} \frac{H \left[ \frac{2K}{\pi} \left( x + \frac{\pi}{2} \right) \right]}{\Theta \left( \frac{2Kx}{\pi} \right)} \\ &= \sqrt{\frac{k'}{k}} \frac{H \left[ \frac{2Kx}{\pi} + K \right]}{\Theta \left( \frac{2Kx}{\pi} \right)}. \end{aligned}$$

Replacing  $\frac{2Kx}{\pi}$  by its value,  $u$ , we have

$$(5) \quad \text{cn } (u, k) = \sqrt{\frac{k'}{k}} \frac{H(u + K)}{\Theta(u)}.$$

Furthermore,

$$(6) \quad \text{dn } (u, k) = \sqrt{k'} \left[ \Pi \right] \frac{1 + 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}$$

gives in the same manner

$$\operatorname{dn} \frac{2Kx}{\pi} = \sqrt{k'} \frac{\Theta \left[ \frac{2K}{\pi} \left( x + \frac{\pi}{2} \right) \right]}{\Theta \left( \frac{2Kx}{\pi} \right)},$$

or

$$(7) \quad \operatorname{dn} (u, k) = \sqrt{k'} \frac{\Theta(u+K)}{\Theta(u)}.$$

If we put

$$(8) \quad H(u+K) = H_1(u),$$

$$(9) \quad \Theta(u+K) = \Theta_1(u),$$

the three elliptic functions can be expressed by the following formulas:

$$(10) \quad \operatorname{sn} (u, k) = \frac{1}{\sqrt{k}} \cdot \frac{H(u)}{\Theta(u)};$$

$$(11) \quad \operatorname{cn} (u, k) = \sqrt{\frac{k'}{k}} \cdot \frac{H_1(u)}{\Theta(u)};$$

$$(12) \quad \operatorname{dn} (u, k) = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}.$$

These functions  $\Theta$  and  $H$  can be expressed in terms of each other. By definition,

$$H(u) = 2C \sqrt{q} \sin \frac{\pi u}{2K} \left[ \Pi \right] \left( 1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h} \right);$$

but

$$\begin{aligned} 1 - 2q^h \cos \frac{\pi u}{K} + q^{2h} &= \left(1 - q^h e^{\frac{\pi u \sqrt{-1}}{K}}\right) \left(1 - q^h e^{-\frac{\pi u \sqrt{-1}}{K}}\right), \\ \sin \frac{\pi u}{2K} &= \frac{e^{\frac{\pi i u}{2K}} - e^{-\frac{\pi i u}{2K}}}{2 \sqrt{-1}} \\ &= e^{\frac{-\pi i u}{2K}} \frac{1 - e^{\frac{\pi i u}{K}}}{2} \sqrt{-1}, \end{aligned}$$

and consequently

$$(13) \quad H(u) = C \sqrt[4]{q} e^{-\frac{\pi i u}{2K}} \sqrt{-1} \left(1 - e^{\frac{\pi i u}{K}}\right) \left(1 - q^2 e^{-\frac{\pi i u}{K}}\right) \left(1 - q^2 e^{\frac{\pi i u}{K}}\right) \dots$$

Now, changing  $u$  into  $u + iK'$ , and remembering that  $e^{\frac{-\pi K'}{K}} = q$ , we have

$$(14) \quad \begin{aligned} H(u + iK') \\ = C q^{-\frac{1}{4}} e^{-\frac{\pi i u}{2K}} \sqrt{-1} \left(1 - q e^{\frac{\pi i u}{K}}\right) \left(1 - q e^{-\frac{\pi i u}{K}}\right) \left(1 - q^3 e^{\frac{\pi i u}{K}}\right) \left(1 - q^3 e^{-\frac{\pi i u}{K}}\right) \dots; \end{aligned}$$

and reuniting the factors two by two, this becomes

$$(15) \quad \begin{aligned} H(u + iK') \\ = C \sqrt{-1} q^{-\frac{1}{4}} e^{-\frac{\pi i u}{2K}} \left(1 - 2q \cos \frac{\pi u}{K} + q^2\right) \left(1 - 2q^3 \cos \frac{\pi u}{K} + q^6\right) \dots; \end{aligned}$$

and finally, according to equation (1),

$$(16) \quad H(u + iK') = \sqrt{-1} q^{-\frac{1}{4}} e^{-\frac{\pi i u}{2K}} \Theta(u).$$

In the same manner, we can get

$$(17) \quad \Theta(u + iK') = \sqrt{-1} q^{-\frac{1}{2}} e^{-\frac{\pi i u}{2K}} H(u).$$

Substituting  $u + 2K$  for  $u$  in equations (1) and (2), we get

$$(18) \quad \Theta(u + 2K) = \Theta(u),$$

$$(19) \quad H(u + 2K) = -H(u),$$

since  $\cos \frac{\pi}{K}(u + 2K) = \cos \frac{\pi u}{K}$  and  $\sin \frac{\pi}{2K}(u + 2K) = -\sin \frac{\pi u}{2K}$ .

The comparison of these four equations with equations (10), (11), and (12) shows the periodicity of the elliptic functions. For example, comparing eqs. (10) and (16) and (17), we see that changing  $u$  into  $u + iK'$  simply multiplies the numerator and denominator of the second member of eq. (10) by the same number, and does not change their ratio.

The addition of  $2K$  changes the sign of the function, but not its value.

We will define  $\Theta_1$  and  $H_1$  as follows:

$$(20) \quad \Theta_1(x) = \Theta(x + K);$$

$$(21) \quad H_1(x) = H(x + K).$$

Hence we get, from equation (17),

$$\begin{aligned} \Theta_1(x + iK') &= \Theta(x + iK' + K) = \Theta(x + K + iK') \\ &= iH(x + K) e^{-\frac{i\pi}{4K}(2x + 2K + iK')} \\ &= iH_1(x) e^{-\frac{i\pi}{4K}(2x + iK')} (-\sqrt{-1}), \end{aligned}$$

$$\text{since} \quad e^{-\frac{i\pi}{2}} = \cos \frac{\pi}{2} - \sqrt{-1} \sin \frac{\pi}{2} = -\sqrt{-1};$$

whence

$$(22) \quad \Theta_1(x + iK') = H_1(x) e^{-\frac{i\pi}{4K}(2x + iK')}.$$

In a similar manner we get

$$(22)^* \quad H_1(x + iK') = \Theta_1(x) e^{-\frac{i\pi}{4K}(2x + iK')}.$$

In eq. (9), Chap. VIII, put  $u = \frac{2Kz}{\pi}$ , and we get

$$(23) \quad \Theta\left(\frac{2Kz}{\pi}\right) = 1 - 2q \cos 2z + 2q^4 \cos 4z - \dots$$

Now, in this equation, changing  $z$  into  $z + \frac{\pi}{2}$ , and observing eq. (20), we get

$$(24) \quad \Theta_1\left(\frac{2Kz}{\pi}\right) = 1 + 2q \cos 2z + 2q^4 \cos 4z + \dots$$

Applying eq. (22) to this, we have

$$\begin{aligned} H_1\left(\frac{2Kz}{\pi}\right) &= \Theta_1\left(\frac{2K}{\pi}\left(z + \frac{\pi iK'}{2K}\right)\right) e^{\frac{\pi i}{4K}\left(\frac{4Kz}{\pi} + iK'\right)} \\ &= \Theta_1\left(\frac{2K}{\pi}\left(z + \frac{\pi iK'}{2K}\right)\right) e^{iz} q^{\frac{1}{4}} \\ &= e^{iz} q^{\frac{1}{4}} \left[ 1 + 2q \cos 2\left(z + \frac{\pi iK'}{2K}\right) + 2q^4 \cos 4\left(z + \frac{\pi iK'}{2K}\right) + \dots \right] \\ &= e^{iz} q^{\frac{1}{4}} \left[ 1 + q \left( e^{2i\left(z + \frac{\pi iK'}{2K}\right)} + e^{-2i\left(z + \frac{\pi iK'}{2K}\right)} \right) \right. \\ &\quad \left. + q^4 \left( e^{4i\left(z + \frac{\pi iK'}{2K}\right)} + e^{-4i\left(z + \frac{\pi iK'}{2K}\right)} \right) + \dots \right] \end{aligned}$$

$$\begin{aligned}
&= e^{iz} q^{\frac{1}{2}} [1 + q(qe^{2iz} + q^{-1}e^{-2iz}) + q^{\frac{1}{2}}(q^2e^{4iz} + q^{-2}e^{-4iz}) + \dots] \\
&= e^{iz} q^{\frac{1}{2}} [1 + q^2e^{2iz} + q^6e^{4iz} + \dots \\
&\quad + e^{-2iz} + q^2e^{-4iz} + \dots] \\
&= q^{\frac{1}{2}} [e^{iz} + q^2e^{3iz} + q^6e^{5iz} + \dots \\
&\quad + e^{-iz} + q^2e^{-3iz} + q^6e^{-5iz} + \dots] \\
&= 2q^{\frac{1}{2}} [\cos z + q^2 \cos 3z + q^6 \cos 5z + \dots];
\end{aligned}$$

whence

$$(25) \quad H_1\left(\frac{2Kz}{\pi}\right) = 2\sqrt[4]{q} \cos z + 2\sqrt[4]{q^9} \cos 3z + 2\sqrt[4]{q^{25}} \cos 5z + \dots$$

In this equation, changing  $z$  into  $z - \frac{\pi}{2}$ , and applying eq. (21), we get

$$(26) \quad H\left(\frac{2Kz}{\pi}\right) = 2\sqrt[4]{q} \sin z - 2\sqrt[4]{q^9} \sin 3z + 2\sqrt[4]{q^{25}} \sin 5z - \dots,$$

since

$$H_1\left(\frac{2Kz}{\pi}\right) = H\left(\frac{2Kz}{\pi} + K\right).$$

We will now determine the constant  $A$  of eq. (8), Chap. VIII, and eqs. (1) and (2) of this chapter. Denote  $A$  by  $f(q)$ , and we have

$$(26)^* \quad [II](1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}) = f(q)\Theta(u).$$

Substituting herein  $u = 0$  and  $u = \frac{K}{2}$ , we have

$$[II](1 - q^{2h-1})^2 = f(q)\Theta(0);$$

$$[II](1 + q^{4h-2}) = f(q)\Theta\left(\frac{K}{2}\right).$$

From eq. (9), Chap. VIII, we get

$$(27) \quad \Theta(0) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \dots;$$

$$(28) \quad \Theta\left(\frac{K}{2}\right) = 1 - 2q^4 + 2q^{16} - 2q^{36} + 2q^{64} - \dots;$$

from which we see that  $\Theta(0)$  is changed into  $\Theta\left(\frac{K}{2}\right)$  when we put  $q^4$  in place of  $q$ .

Whence

$$[II](1 - q^{8h-4})^2 = f(q^4)\Theta\left(\frac{K}{2}\right);$$

and therefore

$$(29) \quad \frac{f(q)}{f(q^4)} = \left[II\right] \frac{1 + q^{4h-2}}{(1 - q^{8h-4})^2} \\ = \left[II\right] \frac{1}{(1 - q^{8h-4})(1 - q^{4h-2})}.$$

Now, the expressions  $4h - 2$ ,  $8h - 4$ , and  $8h$  give the following series of numbers:

$$\begin{array}{cccccccc} 4h - 2, & 2, & 6, & 10, & 14, & 18, & 22, & 26, & 30, & 34; \\ 8h - 4, & 4, & & 12, & & 20, & & 28, & & 36; \\ 8h, & & 8, & & 16, & & 24, & & 32. \end{array}$$

Hence, the three expressions taken together contain all the even numbers, and

$$\left[II\right](1 - q^{8h-4})(1 - q^{4h-2})(1 - q^{8h}) = \left[II\right](1 - q^{2h}).$$

Therefore, multiplying eq. (29) by

$$\left[II\right] \frac{1 - q^{8h}}{1 - q^{8h}},$$

we have

$$\frac{f(q)}{f(q^4)} = \left[II\right] \frac{1 - q^{8h}}{1 - q^{2h}}.$$

Now in this equation, by successive substitutions of  $q^4$  for  $q$ , we get

$$\frac{f(q^4)}{f(q^{16})} = \left[ \Pi \right] \frac{1 - q^{32h}}{1 - q^{8h}};$$

$$\frac{f(q^{16})}{f(q^{64})} = \left[ \Pi \right] \frac{1 - q^{128h}}{1 - q^{32h}};$$

$$\frac{f(q^{64})}{f(q^{256})} = \left[ \Pi \right] \frac{1 - q^{512h}}{1 - q^{128h}};$$

. . . . .

Now  $q$  being less than 1,  $q^n$  tends towards the limit 0 as  $n$  increases, and consequently  $1 - q^n$  tends towards the limit 1. Also, from eq. (8), Chap. VIII, we see that  $f(0) = 1$ . Hence, multiplying the above equations together member by member, we have

$$(30) \quad f(q) = \left[ \Pi \right] \frac{1}{1 - q^{2h}},$$

or

$$(31) \quad A = \frac{1}{(1 - q^2)(1 - q^4)(1 - q^6) \dots}.$$

Substituting this value in equation (8), Chap. VIII, we have, after making  $u = 0$ ,

$$\begin{aligned} (1 - q)^2(1 - q^3)^2(1 - q^5)^2 \dots &= \frac{1 - 2q + 2q^4 - 2q^9 + \dots}{(1 - q^2)(1 - q^4)(1 - q^6) \dots} \\ &= \frac{\Theta(0)}{(1 - q^2)(1 - q^4)(1 - q^6) \dots}. \end{aligned}$$

(See equation (9), Chap. VIII.)

Transposing one of the series of products from the left-hand member, we get

$$(1 - q)(1 - q^3) \dots = \frac{\Theta(0)}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4) \dots}.$$

Introducing on both sides of the equation the factors  $1 - q^2$ ,  $1 - q^4$ ,  $1 - q^6$ , etc., we get

$$\begin{aligned} & (1 - q)(1 - q^2)(1 - q^3)(1 - q^4) \dots \\ &= \Theta(0) \frac{1 - q^2}{1 - q} \cdot \frac{1 - q^4}{1 - q^2} \cdot \frac{1 - q^6}{1 - q^3} \cdot \frac{1 - q^8}{1 - q^4} \dots \\ &= \Theta(0)(1 + q)(1 + q^2)(1 + q^3) \dots; \end{aligned}$$

whence

$$(32) \quad \Theta(0) = \frac{(1 - q)(1 - q^2)(1 - q^3)}{(1 + q)(1 + q^2)(1 + q^3)}.$$

Resuming equation (20), Chap. VII, and dividing both members of the equation by  $u$ , we have

$$\frac{\operatorname{sn} u}{u} = \frac{2\sqrt[4]{q}}{\sqrt{k}} \cdot \frac{\sin \frac{\pi u}{2K}}{u} \left[ \Pi \right] \frac{1 - 2q^{2h} \cos \frac{\pi u}{K} + q^{4h}}{1 - 2q^{2h-1} \cos \frac{\pi u}{K} + q^{4h-2}}.$$

This, for  $u = 0$ , since the limiting value of  $\frac{\operatorname{sn} u}{u}$  for  $u = 0$  is 1,

and of  $\frac{\sin \frac{\pi x}{2K}}{u}$  for  $x = 0$  is  $\frac{\pi}{2K}$ , becomes

$$1 = \frac{\sqrt[4]{q}}{\sqrt{k}} \cdot \frac{\pi}{K} \cdot \frac{(1 - q^2)(1 - q^4)(1 - q^6) \dots}{(1 - q)^2(1 - q^3)^2(1 - q^5)^2 \dots},$$

or

$$(33) \quad \frac{\sqrt{k}K}{\pi \sqrt[4]{q}} = \left[ \frac{(1 - q^2)(1 - q^4)(1 - q^6) \dots}{(1 - q)(1 - q^3)(1 - q^5) \dots} \right]^2.$$

Further, from equation (21), Chap. VII, for  $u = 0$ , we have

$$(34) \quad \frac{\sqrt{k}}{2\sqrt{k'}\sqrt[4]{q}} = \left[ \frac{(1 + q^2)(1 + q^4)(1 + q^6) \dots}{(1 - q)(1 - q^3)(1 - q^5) \dots} \right]^2.$$

The quotient of these two equations gives

$$(35) \quad \frac{2\sqrt{k'}K}{\pi} = \left[ \frac{(1-q^2)(1-q^4)(1-q^6)\dots}{(1+q^2)(1+q^4)(1+q^6)\dots} \right]^2;$$

or, substituting the value of  $\sqrt{k'}$  from eqs. (18) and (19), Chap. VII,

$$(36) \quad \frac{2k'K}{\pi} = \left[ \frac{(1-q)(1-q^3)(1-q^5)\dots}{(1+q)(1+q^3)(1+q^5)\dots} \right]^2.$$

Comparing this with equation (32), we easily get

$$(37) \quad \Theta(0) = \sqrt{\frac{2k'K}{\pi}}.$$

From equation (9), Chap. VIII, making  $u = K$ , we get

$$(38) \quad \Theta(K) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots$$

Making  $z = 0$  in equation (24), Chap. IX, we have

$$(39) \quad \Theta_1(0) = 1 + 2q + 2q^4 + 2q^9 + \dots$$

This might also have been derived from eq. (38) by observing that

$$\Theta(0 + K) = \Theta_1(0) = \Theta(K).$$

Knowing  $\Theta(0)$ , it is easy to deduce  $\Theta(K)$  and  $H(K)$

From equation (7) we have

$$\operatorname{dn} u = \sqrt{k'} \frac{\Theta(u + K)}{\Theta(u)}.$$

Making  $u = 0$ , we have, since  $\operatorname{dn}(0) = 1$ ,

$$(40) \quad \Theta(K) = \frac{\Theta(0)}{\sqrt{k'}}.$$

From equation (5) we get, in the same manner,

$$(41) \quad H(K) = \sqrt{\frac{k'}{k}} \Theta(0).$$

From eq. (12), Chap. IX, we have

$$(41)^* \quad \operatorname{dn} u = \sqrt{1 - k^2 \sin^2 \phi} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)};$$

and putting  $x = \frac{\pi u}{2K}$ , we have

$$(42) \quad \frac{\operatorname{dn} u}{\sqrt{k'}} = \frac{1 + 2q \cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \dots}{1 - 2q \cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \dots}.$$

Putting

$$(42)^* \quad \frac{\operatorname{dn} u}{\sqrt{k'}} = \cot \gamma,$$

we have

$$\frac{\cot \gamma - 1}{\cot \gamma + 1} = \tan(45^\circ - \gamma) = 2q \frac{\cos 2x + q^4(4 \cos^3 2x - 3 \cos 2x) + \dots}{1 + q^4(4 \cos^2 2x - 2)};$$

whence

$$(43) \quad \cos 2x = \frac{\tan(45^\circ - \gamma)[1 + q^4(4 \cos^2 2x - 2)]}{2q} \\ - q^8(4 \cos^3 2x - 3 \cos 2x) - \dots,$$

and approximately,

$$(44) \quad \cos 2x = \frac{\tan(45^\circ - \gamma)}{2q}.$$

From equations (37) and (40), Chap. IX, we have

$$(45) \quad x = \frac{u}{\Theta^2(K)};$$

whence

$$(46) \quad u = x\Theta^2(K).$$

## CHAPTER X.

### ELLIPTIC INTEGRALS OF THE SECOND ORDER.

FROM Chap. I, equation (19), we have

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi} \cdot d\phi = \int_0^\phi \Delta\phi \cdot d\phi.$$

From this we have

$$E(\phi) + E(\psi) = \int_0^\phi \Delta\phi \cdot d\phi + \int_0^\psi \Delta\phi \cdot d\phi.$$

Put

$$(1) \quad E\phi + E\psi = S.$$

Differentiating, we get

$$(2) \quad \Delta\phi \cdot d\phi + \Delta\psi \cdot d\psi = dS.$$

But we have, Chap. II, equation (2),

$$\frac{d\phi}{\Delta\phi} + \frac{d\psi}{\Delta\psi} = 0,$$

or

$$(3) \quad \Delta\psi \cdot d\phi + \Delta\phi \cdot d\psi = 0.$$

Adding equations (2) and (3), we get

$$(4) \quad (\Delta\phi + \Delta\psi)(d\phi + d\psi) = dS.$$

Substituting  $\cos \mu$  from eq. (5), in eq. (5)\*, Chap. II, we get

$$(5) \quad \begin{cases} \Delta\phi = \frac{\sin \phi \cos \psi \Delta\mu + \cos \phi \sin \psi}{\sin \mu}, \\ \Delta\psi = \frac{\sin \psi \cos \phi \Delta\mu + \cos \psi \sin \phi}{\sin \mu}; \end{cases}$$

whence

$$(6) \quad \Delta\phi \pm \Delta\psi = \frac{\Delta\mu \pm 1}{\sin \mu} \sin(\phi \pm \psi).$$

Substituting in equation (4), we have

$$(7) \quad \begin{aligned} dS &= \frac{\Delta\mu + 1}{\sin \mu} \sin(\phi + \psi) d(\phi + \psi) \\ &= - \frac{\Delta\mu + 1}{\sin \mu} d \cos(\phi + \psi). \end{aligned}$$

Integrating equation (7), we have

$$E\phi + E\psi = \frac{\Delta\mu + 1}{\sin \mu} [C - \cos(\phi + \psi)].$$

The constant of integration,  $C$ , is determined by making  $\phi = 0$ ; in this case  $\psi = \mu$ ,  $E\phi = 0$ ,  $E\psi = E\mu$ , and  $S = E\mu$ ; whence

$$E\mu = \frac{\Delta\mu + 1}{\sin \mu} (C - \cos \mu),$$

and by subtraction,

$$E\phi + E\psi - E\mu = \frac{\Delta\mu + 1}{\sin \mu} (\cos \mu - \cos \phi \cos \psi + \sin \phi \sin \psi).$$

But, Chap. II, eq. (5),

$$\cos \mu - \cos \phi \cos \psi = - \sin \phi \sin \psi \Delta\mu;$$

whence

$$E\phi + E\psi - E\mu = \frac{1 - \Delta^2\mu}{\sin \mu} \sin \phi \sin \psi$$

whence

$$(8) \quad E\phi + E\psi = E\mu + k^2 \sin \phi \sin \psi \sin \mu.$$

When  $\phi = \psi$ , we have

$$(9) \quad E\mu = 2E\phi - k^2 \sin^2 \phi \sin \mu.$$

But in that case

$$(10) \quad \cos \mu = \cos^2 \phi - \sin^2 \phi \Delta\mu;$$

whence

$$(11) \quad \sin \phi = \sqrt{\frac{1 - \cos \mu}{1 + \Delta\mu}}.$$

Let  $\phi, \phi_{\frac{1}{2}}, \phi_{\frac{1}{4}}$ , etc., be such values as will satisfy the equations

$$(12) \quad \begin{aligned} E\phi &= 2E\phi_{\frac{1}{2}} - k^2 \sin^2 \phi_{\frac{1}{2}} \sin \phi, \\ E\phi_{\frac{1}{2}} &= 2E\phi_{\frac{1}{4}} - k^2 \sin^2 \phi_{\frac{1}{4}} \sin \phi_{\frac{1}{2}}, \\ &\text{etc.} \qquad \qquad \qquad \text{etc.} \end{aligned}$$

Assume an auxiliary angle  $\gamma$ , such that

$$(13) \quad \sin \gamma = k \sin \phi;$$

whence

$$\Delta\phi = \cos \gamma,$$

and Chap. IV, eq. (24),

$$(14) \quad \sin \phi_{\frac{1}{2}} = \frac{\sin \frac{1}{2}\phi}{\cos \frac{1}{2}\gamma}.$$



Substituting in eq. (16) the numerical values derived from equations (15) and (17), we are enabled to determine the value of  $E\phi$ .

Landen's Transformation can also be applied to Elliptic Integrals of this class.

From eq. (11), Chap. IV, we get, by easy transformation,

$$(18) \quad \sin^2 2\phi = \sin^2 \phi_1 (1 + k_0 + 2k_0 \cos 2\phi).$$

From this we easily get

$$\begin{aligned} 2k_0 \cos 2\phi \sin^2 \phi_1 &= \sin^2 2\phi - \sin^2 \phi_1 - k_0^2 \sin^2 \phi_1 \\ &= 1 - \cos^2 2\phi - \sin^2 \phi_1 - k_0^2 \sin^2 \phi_1 \\ &= \Delta^2 k_0 \phi_1 - \sin^2 \phi_1 - \cos^2 2\phi; \end{aligned}$$

whence

$$\cos^2 2\phi + 2k_0 \sin^2 \phi_1 \cos 2\phi = \Delta^2 k_0 \phi_1 - \sin^2 \phi_1;$$

and from this,

$$\begin{aligned} \cos 2\phi &= -k_0 \sin^2 \phi_1 \pm \sqrt{\Delta^2 k_0 \phi_1 - \sin^2 \phi_1 + k_0^2 \sin^4 \phi_1} \\ (19) \quad &= \cos \phi_1 \Delta k_0 \phi_1 - k_0 \sin^2 \phi_1; \end{aligned}$$

whence, also,

$$\begin{aligned} 1 - \cos^2 2\phi &= 1 - \cos^2 \phi_1 \Delta^2 \phi_1 + 2k \sin^2 \phi_1 \cos \phi_1 \Delta k_0 \phi_1 - k_0^2 \sin^4 \phi_1 \\ &= \sin^2 \phi_1 (1 + k_0^2 \cos^2 \phi_1 + 2k_0 \cos \phi_1 \Delta k_0 \phi_1 - k_0^2 \sin^2 \phi_1) \end{aligned}$$

and

$$(20) \quad \sin 2\phi = \sin \phi_1 (\Delta k_0 \phi_1 + k_0 \cos \phi_1).$$

Differentiating equation (19), we get

$$2 \sin 2\phi \frac{d\phi}{d\phi_1} = \sin \phi_1 \frac{(k_0 \cos \phi_1 + \Delta k_0 \phi_1)^2}{\Delta k_0 \phi_1}.$$

Dividing this by equation (20), we have

$$\frac{2d\phi}{d\phi_1} = \frac{k_0 \cos \phi_1 + \Delta k_0 \phi_1}{\Delta k_0 \phi_1}.$$

But from (19), and eq. (6), Chap. IV,

$$\begin{aligned} k^2 \sin^2 \phi &= \frac{k^2 (1 - \cos 2\phi)}{2} \\ &= \frac{2k_0}{(1 + k_0)^2} \{1 + k_0 \sin^2 \phi_1 - \cos \phi_1 \Delta k_0 \phi_1\}; \end{aligned}$$

whence

$$\Delta k \phi = \frac{\Delta k_0 \phi_1 + k_0 \cos \phi_1}{1 + k_0},$$

and

$$2\Delta k \phi \cdot \frac{d\phi}{d\phi_1} = \frac{(k_0 \cos \phi_1 + \Delta k_0 \phi_1)^2}{(1 + k_0) \Delta k_0 \phi_1},$$

and

$$d\phi \Delta k \phi = \frac{d\phi_1}{\Delta k_0 \phi_1} \cdot \frac{(k_0 \cos \phi_1 + \Delta k_0 \phi_1)^2}{2(1 + k_0)}.$$

This gives immediately, by integration,

$$\begin{aligned} Ek\phi &= \frac{1}{2(1 + k_0)} \int \frac{d\phi_1}{\Delta k_0 \phi_1} \{k_0 \cos \phi_1 + \Delta k_0 \phi_1\}^2 \\ &= \frac{1}{2(1 + k_0)} \int \frac{d\phi_1}{\Delta k_0 \phi_1} \{2\Delta^2 k_0 \phi_1 + 2k_0 \cos \phi_1 \Delta k_0 \phi_1 - k_1'^2\} \\ (21) \quad &= \frac{Ek_0 \phi_1}{1 + k_0} + \frac{k_0 \sin \phi_1}{1 + k_0} - \frac{1}{2}(1 - k_0) Fk_0 \phi_1. \end{aligned}$$

Thus the value of  $Ek\phi$  is made to depend upon  $Ek_0 \phi_1$  (containing a smaller modulus and a larger amplitude), and upon the integral of the first class,  $Fk_0 \phi_1$ ;  $k_0$ ,  $\phi_1$ , etc., being determined by the formulæ (6) to (12) of Chap. IV.

By successive applications of equation (21),  $Ek\phi$  may be made to depend ultimately upon  $Ek_{0n} \phi_n$ , where  $k_{0n}$  approximates to zero and  $Ek_{0n} \phi_n$  to  $\phi_n$ .

Or, by reversing, it may be made to depend upon  $Ek_n \phi_{0n}$ , where  $k_n$  approximates to unity and  $Ek_n \phi_{0n}$  to  $-\cos \phi_{0n}$ .

To facilitate this, assume

$$Gk\phi = Ek\phi - Fk\phi.$$

Subtracting from equation (21) the equation

$$Fk\phi = \frac{1+k_0}{2}Fk_0\phi_1 \text{ (see eq. (13), Chap. IV),}$$

we have

$$Gk\phi = \frac{1}{1+k_0}(Gk_0\phi_1 + k_0 \sin \phi_1 - k_0 Fk_0\phi_1).$$

Repeated applications of this give

$$Gk_0\phi_1 = \frac{1}{1+k_{00}}(Gk_{00}\phi_2 + k_{00} \sin \phi_2 - k_{00} Fk_{00}\phi_2),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Gk_{0(n-1)}\phi_{n-1} = \frac{1}{1+k_{0n}}(Gk_{0n}\phi_n + k_{0n} \sin \phi_n - k_{0n} Fk_{0n}\phi_n).$$

Whence

$$(22) \quad Gk\phi = \sum_n \left\{ \frac{k_{0n}(\sin \phi_n - Fk_{0n}\phi_n)}{\left[\frac{1}{n}\right](1+k_{0n})} \right\} + \frac{Gk_{0n}\phi_n}{\left[\frac{1}{n}\right](1+k_{0n})}.$$

But since (compare eq. (13), Chap. IV)

$$Fk\phi = \frac{Fk_{0n}\phi_n \left[\frac{1}{n}\right](1+k_{0n})}{2^n},$$

or

$$(23) \quad \frac{Fk_{0n}\phi_n}{\left[\frac{1}{n}\right](1+k_{0n})} = \frac{2^n Fk\phi}{\left[\frac{1}{n}\right](1+k_{0n})^2};$$

and since, also, (compare eq. (6), Chap. IV,)

$$\frac{k_{0(n-1)}^2}{k_{0n}} = \frac{2^2}{(1 + k_{0n})^2},$$

we have

$$\begin{aligned} (24) \quad \frac{2^n k_{0n}}{\left[\frac{1}{II}\right]_n (1 + k_{0n})^2} &= \frac{k_{0n}}{2^n} \left[\frac{1}{II}\right]_n \frac{k_{0(n-1)}^2}{k_{0n}} \\ &= \frac{k_{0n}}{2^n} \left[\frac{1}{II}\right]_n \frac{k_{0(n-1)}}{k_{0n}} \left[\frac{1}{II}\right]_n k_{0(n-1)} \\ &= \frac{k_{0n}}{2^n} \cdot \frac{k}{k_0} \cdot \frac{k_0}{k_{00}} \cdots \frac{k_{0(n-1)}}{k_{0n}} \cdot k \left[\frac{2}{II}\right]_n k_{0(n-1)} \\ &= \frac{k^2}{2^n} \left[\frac{2}{II}\right]_n k_{0(n-1)}. \end{aligned}$$

Substituting these values in equation (22), and neglecting the term containing  $Gk_{0n}\phi_n$  since, carried to its limiting value,

$$\begin{aligned} Gk_{0n}\phi_n &= Ek_{0n}\phi_n - Fk_{0n}\phi_n \\ &= \phi_n - \phi_n = 0, \quad (n = \text{limiting value,}) \end{aligned}$$

we have

$$\begin{aligned} (25) \quad Gk\phi &= \sum_n \left\{ \frac{k \sqrt{k_{0n}} \sin \phi_n \left[\frac{2}{II}\right]_n \sqrt{k_{0(n-1)}} - k^2 \left[\frac{2}{II}\right]_n k_{0(n-1)}}{2^n} \right\} \\ &= k \left[ \frac{\sqrt{k_0}}{2} \sin \phi_1 + \frac{\sqrt{k_0 k_{00}}}{2^2} \sin \phi_2 + \frac{\sqrt{k_0 k_{00} k_{03}}}{2^3} \sin \phi_3 + \dots \right] \\ &\quad - \frac{k^2}{2} \left[ 1 + \frac{k_0}{2} + \frac{k_0 k_{00}}{2^2} + \frac{k_0 k_{00} k_{03}}{2^3} + \dots \right]; \end{aligned}$$

whence

$$\begin{aligned} (26) \quad Ek\phi &= Fk\phi \left[ 1 - \frac{k^2}{2} \left( 1 + \frac{k_0}{2} + \frac{k_0 k_{00}}{2^2} + \dots \right) \right] \\ &+ k \left[ \frac{\sqrt{k_0}}{2} \sin \phi_1 + \frac{\sqrt{k_0 k_{00}}}{2^2} \sin \phi_2 + \frac{\sqrt{k_0 k_{00} k_{03}}}{2^3} \sin \phi_3 + \dots \right]. \end{aligned}$$

From eq. (3), Chap. V, we see that when  $\phi = \frac{\pi}{2}$ ,

$$\phi_n = 2^{n-1}\pi.$$

Substituting these values in equation (26), we have for a complete Elliptic Integral of the second class,

$$(27) \quad E\left(k, \frac{\pi}{2}\right) = F\left(k, \frac{\pi}{2}\right) \left[ 1 - \frac{k^2}{2} \left( 1 + \frac{k_0}{2} + \frac{k_0 k_{00}}{2^2} + \frac{k_0 k_{00} k_{03}}{2^3} + \dots \right) \right].$$

In a similar manner we could have found the formula for  $E(k, \phi)$  in terms of an increasing modulus, viz.,

$$(28) \quad E(k, \phi) = F(k, \phi) \left[ 1 + k \left( 1 + \frac{2}{k_1} + \frac{2^2}{k_1 k_2} + \frac{2^3}{k_1 k_2 k_3} + \dots \right. \right. \\ \left. \left. + \frac{2^{n-2}}{k_1 k_2 \dots k_{n-2}} - \frac{2^{n-1}}{k_1 k_2 \dots k_{n-1}} \right) \right] \\ - k \left[ \sin \phi + \frac{2}{\sqrt{k}} \sin \phi_1 + \frac{2^2}{\sqrt{k k_1}} \sin \phi_2 + \dots \right. \\ \left. + \frac{2^{n-1}}{\sqrt{k k_1 \dots k_{n-2}}} \sin \phi_{n-1} - \frac{2^n}{\sqrt{k k_1 \dots k_{n-1}}} \sin \phi_n \right].$$

## CHAPTER XI.

### ELLIPTIC INTEGRALS OF THE THIRD ORDER.

THE Elliptic Integral of the third order is

$$(1) \quad \Pi(n, k, \phi) = \int_0^\phi \frac{d\phi}{(1 + n \sin^2 \phi) \Delta \phi}.$$

Put

$$(2) \quad \Pi(\phi) + \Pi(\psi) = S;$$

whence we have immediately

$$(3) \quad dS = \frac{d\phi}{(1 + n \sin^2 \phi) \Delta \phi} + \frac{d\psi}{(1 + n \sin^2 \psi) \Delta \psi}.$$

But, eq. (2), Chap. II,

$$(4) \quad \frac{d\phi}{\Delta \phi} + \frac{d\psi}{\Delta \psi} = 0;$$

whence

$$(5) \quad \begin{aligned} dS &= \left( \frac{1}{1 + n \sin^2 \phi} - \frac{1}{1 + n \sin^2 \psi} \right) \frac{d\phi}{\Delta \phi} \\ &= \frac{n(\sin^2 \psi - \sin^2 \phi)}{(1 + n \sin^2 \phi)(1 + n \sin^2 \psi)} \cdot \frac{d\phi}{\Delta \phi}. \end{aligned}$$

From equation (8), Chap. X, we get by differentiation, since  $\sigma$  (or  $\mu$ ) is constant,

$$\Delta \phi \cdot d\phi + \Delta \psi \cdot d\psi = k^2 \sin \sigma d(\sin \phi \sin \psi),$$

or, from equation (3),

$$(\sin^2 \psi - \sin^2 \phi) \frac{d\phi}{\Delta \phi} = \sin \sigma d(\sin \phi \sin \psi).$$

This, introduced into equation (5), gives

$$dS = \frac{n \sin \sigma d(\sin \phi \sin \psi)}{1 + n(\sin^2 \phi + \sin^2 \psi) + n^2 \sin^2 \phi \sin^2 \psi}.$$

Put

$$\sin \phi \sin \psi = q, \quad \sin^2 \phi + \sin^2 \psi = p;$$

whence

$$(6) \quad dS = \frac{n \sin \sigma dq}{1 + np + n^2 p^2}.$$

From equation (5), Chap. II, we have

$$\cos \sigma = \cos \phi \cos \psi - \sin \phi \sin \psi \Delta \sigma,$$

from which we easily get

$$\begin{aligned} (\cos \sigma + q \Delta \sigma)^2 &= \cos^2 \phi \cos^2 \psi \\ &= (1 - \sin^2 \phi)(1 - \sin^2 \psi) \\ &= 1 - p + q^2, \end{aligned}$$

and thence

$$\begin{aligned} p &= 1 + q^2 - (\cos \sigma + q \Delta \sigma)^2 \\ &= \sin^2 \sigma - 2 \cos \sigma \Delta \sigma q + k^2 \sin^2 \sigma \cdot q^2. \end{aligned}$$

This, substituted in eq. (6), gives

$$\begin{aligned} dS &= \frac{n \sin \sigma dq}{1 + n \sin^2 \sigma - 2n \cos \sigma \Delta \sigma q + n(n + k^2 \sin^2 \sigma)q^2} \\ &= \frac{n \sin \sigma dq}{A - 2Bq + Cq^2}, \end{aligned}$$

where

$$\begin{aligned} A &= 1 + n \sin^2 \sigma, \\ B &= n \cos \sigma \Delta \sigma, \\ C &= nk^2 \sin^2 \sigma + n^2. \end{aligned}$$

From this we get

$$S = n \sin \sigma \int \frac{dq}{A - 2Bq + Cq^2} + \text{Const.}$$

In order to determine the constant of integration we must observe that for  $\phi = 0$ ,  $\psi = \sigma$  and  $q = 0$ ; whence

$$\Pi\sigma = n \sin \sigma \int_{q=0} \frac{dq}{A - 2Bq + Cq^2} + \text{Const.};$$

whence

$$S = \Pi\sigma + n \sin \sigma \int_0^q \frac{dq}{A - 2Bq + Cq^2},$$

or

$$(7) \quad \Pi\phi + \Pi\psi = \Pi\sigma + n \sin \sigma \int_0^q \frac{dq}{A - 2Bq + Cq^2}.$$

But we have

$$\begin{aligned} dS &= \frac{CMdq}{AC - B^2 + (Cq - B)^2} \\ &= \frac{CM}{AC - B^2} \cdot \frac{dq}{1 + \left(\frac{Cq - B}{\sqrt{AC - B^2}}\right)^2} \\ &= \frac{M}{\sqrt{AC - B^2}} \cdot \frac{\frac{Cdq}{\sqrt{AC - B^2}}}{1 + \left(\frac{Cq - B}{\sqrt{AC - B^2}}\right)^2}, \end{aligned}$$

where  $M = n \sin \sigma$ .

The integral of the second member is

$$\frac{M}{\sqrt{AC - B^2}} \tan^{-1} \frac{Cq - B}{\sqrt{AC - B^2}};$$

whence

$$\int_0^q dS = S_1 = \frac{M}{\sqrt{AC-B^2}} \left[ \tan^{-1} \frac{Cq-B}{\sqrt{AC-B^2}} + \tan^{-1} \frac{B}{\sqrt{AC-B^2}} \right];$$

or, since

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy},$$

$$S_1 = \frac{M}{\sqrt{AC-B^2}} \tan^{-1} q \frac{\sqrt{AC-B^2}}{A-Bq}.$$

Substituting the values of  $A$ ,  $B$ ,  $C$  and  $M$ , we have

$$\begin{aligned} AC-B^2 &= n(1+n-\Delta^2\sigma)(1+n\sin^2\sigma) - n^2\cos^2\sigma\Delta^2\sigma \\ &= n(1+n-\Delta^2\sigma + n(1+n)\sin^2\sigma - n\Delta^2\sigma) \\ &= n(1+n)(1-\Delta^2\sigma + n\sin^2\sigma) \\ &= n(1+n)(k^2+n)\sin^2\sigma; \end{aligned}$$

and putting

$$\frac{(1+n)(k^2+n)}{n} = \Omega,$$

we have

$$\sqrt{AC-B^2} = n\sqrt{\Omega}\sin\sigma.$$

Substituting these values in eq. (7), we have

$$\begin{aligned} &\Pi(n, k, \phi) + \Pi(n, k, \psi) - \Pi(n, k, \sigma) = S_1 \\ &= \frac{1}{\sqrt{\Omega}} \tan^{-1} \frac{n\sqrt{\Omega}\sin\phi\sin\psi\sin\sigma}{1+n\sin^2\sigma - n\sin\phi\sin\psi\cos\sigma\Delta\sigma}. \end{aligned}$$

## CHAPTER XII.

### NUMERICAL CALCULATIONS. $q$ .

#### CALCULATION OF THE VALUE OF $q$ .

FROM eq. (7), Chap. IX, we have

$$\operatorname{dn} u = \sqrt{k'} \frac{\Theta(u + K)}{\Theta(u)};$$

whence, eq. (9), Chap. IV, eqs. (27) and (39), Chap. IX,

$$\begin{aligned} (1) \quad \sqrt{\cos \theta} &= \frac{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \dots}{1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots} \\ &= 1 - 4q + 8q^2 - 16q^3 + 32q^4 - 56q^5 + \dots \end{aligned}$$

The first five terms of this series can be represented by

$$\sqrt{\cos \theta} = \frac{1 - 2q}{1 + 2q}.$$

From this we get

$$(2) \quad q = \frac{1}{2} \cdot \frac{1 - \sqrt{\cos \theta}}{1 + \sqrt{\cos \theta}},$$

which is exact up to the term containing  $q^5$ .

Or we can deduce a more exact formula as follows: From eq. (1),

$$\begin{aligned} \frac{1 + \sqrt{\cos \theta}}{1 - \sqrt{\cos \theta}} &= \frac{\sqrt{1 + \tan^2 \frac{1}{2}\theta} + \sqrt{1 - \tan^2 \frac{1}{2}\theta}}{\sqrt{1 + \tan^2 \frac{1}{2}\theta} - \sqrt{1 - \tan^2 \frac{1}{2}\theta}} \\ &= \frac{1 + 2q^4 + 2q^{16} + \dots}{2q + 2q^9 + 2q^{25} + \dots}; \end{aligned}$$

whence, by the method of indeterminate coefficients,

$$(3) \quad q = \frac{1}{4} \tan^2 \frac{\theta}{2} + \frac{1}{16} \tan^4 \frac{\theta}{2} + \frac{57}{512} \tan^6 \frac{\theta}{2} + \frac{45}{2048} \tan^8 \frac{\theta}{2} + \dots,$$

or

$$\log q = 2 \log \tan \frac{\theta}{2} - \log 4 +$$

$$\log \left( 1 + \frac{1}{4} \tan^2 \frac{\theta}{2} + \frac{17}{128} \tan^4 \frac{\theta}{2} + \frac{45}{512} \tan^6 \frac{\theta}{2} + \dots \right)$$

$$(4) \quad = 2 \log \tan \frac{\theta}{2} - \log 4 +$$

$$M \left( \frac{1}{4} \tan^2 \frac{\theta}{2} + \frac{17}{128} \tan^4 \frac{\theta}{2} + \frac{23}{384} \tan^6 \frac{\theta}{2} + \dots \right),$$

$M$  being the modulus of the common system of logarithms.

Put

$$(5) \quad \log q = 2 \log \tan \frac{\theta}{2} + 9.397940 + a \tan^2 \frac{\theta}{2} +$$

$$b \tan^4 \frac{\theta}{2} + c \tan^6 \frac{\theta}{2} + \dots,$$

in which

$$\log a = 9.0357243;$$

$$\log b = 8.64452;$$

$$\log c = 8.41518;$$

$$\log d = 8.25283.$$

EXAMPLE. Let  $k' = \cos 10^\circ 23' 46''$ . To find  $q$ .

$$4 \log \tan \frac{\theta}{2} = 5.835 \qquad 2 \log \tan \frac{\theta}{2} = 7.9176842$$

$$\log a = 9.036 \qquad 9.3979400$$

$$\underline{4.871} \qquad \underline{74}$$

$$\log q = 7.3156316$$

$$a \tan^2 \frac{\theta}{2} = 0.0000074$$

When  $\theta$  approaches  $90^\circ$ ,  $\tan \frac{\theta}{2}$  differs little from unity, and the series in eq. (5) is not very converging, but  $q$  can be calculated by means of eq. (6), Chap. VII, viz.,

$$q = e^{-\frac{\pi K'}{K}}, \quad q' = e^{-\frac{\pi K}{K'}}.$$

By comparing these equations with eqs. (6) and (9), Chap. IV, we see that if

$$q = f(k) = f(\theta),$$

then

$$q' = f(k') = f(90^\circ - \theta).$$

Therefore, having  $\theta$ , we can from its complement,  $90^\circ - \theta$ , find  $q'$  by eq. (5), and thence  $q$  by the following process. We have

$$\frac{1}{q} = e^{\frac{\pi K'}{K}}, \quad \frac{1}{q'} = e^{\frac{\pi K}{K'}};$$

whence

$$\log \frac{1}{q} \log \frac{1}{q'} = M^2 \pi^2 = 1.8615228,$$

$$(6) \quad \log \log \frac{1}{q} + \log \log \frac{1}{q'} = 0.2698684,$$

by which we can deduce  $q$  from  $q'$ .

EXAMPLE. Let  $\theta = 79^\circ 36' 14''$ . To find  $q$ .

$$90^\circ - \theta = 10^\circ 23' 46''.$$

By eq. (5) we get

$$\log q' = 7.3156316, \quad \log \frac{1}{q'} = 2.6843684,$$

$$\text{and } \log \log \frac{1}{q'} = .4288421;$$

and by eq. (6),

$$\log \log \frac{1}{q} = 9.8410263;$$

whence

$$\log q = \bar{1}.3065321.$$

When  $k' = k = \cos 45^\circ = \frac{1}{2} \sqrt{2}$ , eq. (6) becomes

$$(7) \quad \log \frac{1}{q} = M\pi = 1.3643763; \quad (k = k';)$$

whence

$$\begin{aligned} \log q &= \bar{2}.6356237, \\ q &= 0.0432138. \end{aligned} \quad (k = k'.)$$

EXAMPLE. Given  $\theta = 10^\circ 23' 46''$ . Find  $q$ .

$$\text{Ans. } \log q = 7.3156316.$$

EXAMPLE. Given  $\theta = 82^\circ 45'$ . Find  $q$ .

$$\text{Ans. } \log q = 9.37919.$$

## CHAPTER XIII.

### NUMERICAL CALCULATIONS. $K$ .

#### CALCULATION OF THE VALUE OF $K$ .

WE have already found from eq. (37), Chap. IX,

$$(1) \quad \Theta(0) = \sqrt{\frac{2\bar{k}'\bar{K}}{\pi}},$$

and from eq. (40), same chapter,

$$(2) \quad \Theta(K) = \frac{\Theta(0)}{\sqrt{\bar{k}'}} = \sqrt{\frac{2\bar{K}}{\pi}}.$$

But, eqs. (38) and (27), Chap. IX,

$$\Theta(K) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots,$$

$$\Theta(0) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \dots;$$

whence, eq. (2),

$$(3) \quad K = \frac{\pi}{2} (1 + 2q + 2q^4 + 2q^9 + \dots)^2.$$

By adding eqs. (1) and (2) we get

$$\Theta(0) + \Theta(K) = \sqrt{\frac{2\bar{K}}{\pi}} (1 + \sqrt{\bar{k}'});$$

whence

$$\begin{aligned} K &= \frac{\pi}{2} \left( \frac{\Theta(0) + \Theta(K)}{1 + \sqrt{\bar{k}'}} \right)^2 \\ &= \frac{\pi}{2} \left[ \frac{2(1 + 2q^4 + 2q^{16} + \dots)}{1 + \sqrt{\bar{k}'}} \right]^2 \end{aligned}$$

$$(4) \quad = \frac{\pi}{2} \left( \frac{2}{1 + \sqrt{k'}} \right)^2 (1 + 2q^4 + 2q^{16} + \dots)^2.$$

EXAMPLE. Let  $k = \sin \theta = \sin 19^\circ 30'$ . Required  $K$ .

*First Method.* By eq. (3).

By eq. (5), Chap. XII, we find  $\log q = 8.6356236$ . Applying eq. (3), using only two terms of the series, we have

$$\begin{aligned} 1 + 2q &= 1.0147662 \\ \log(1 + 2q) &= 0.0063660 \\ 2 \log(1 + 2q) &= 0.0127320 \\ \log \frac{\pi}{2} &= 0.1961199 \\ \hline \log K &= 0.2088519 \\ K &= 1.615101 \end{aligned}$$

*Second Method.* By eq. (4).

Equation (4) may be written, neglecting  $q^4$ ,

$$K = \frac{\pi}{2} \left( \frac{1 + \sqrt{\cos \theta}}{2} \right)^{-2};$$

whence

$$\begin{aligned} \log \cos \theta &= 9.9743466, \\ \log \sqrt{\cos \theta} &= 9.9871733, \\ 1 + \sqrt{\cos \theta} &= 1.9708973, \\ \frac{1 + \sqrt{\cos \theta}}{2} &= 0.98544865; \end{aligned}$$

and

$$\begin{aligned} \log K &= 0.2088519, \\ K &= 1.615101, \end{aligned}$$

the same result as above.

*Third Method.* By eq. (7), Chap. V.

|                                                                                                     |                                            |
|-----------------------------------------------------------------------------------------------------|--------------------------------------------|
| $\theta = 19^\circ 30'$                                                                             | $\theta_0 = 1^\circ 41' 31''.1$            |
| $\frac{1}{2}\theta = 9^\circ 45'$                                                                   | $\frac{1}{2}\theta_0 = 0^\circ 50' 45''.5$ |
| $\log \tan \frac{1}{2}\theta = 9.235103$                                                            |                                            |
| $\log \cos \frac{1}{2}\theta = 9.993681$                                                            | $\log \cos \frac{1}{2}\theta_0 = 9.999953$ |
| $\log \tan^2 \frac{1}{2}\theta \left. \vphantom{\log \tan^2 \frac{1}{2}\theta} \right\} = 8.470206$ |                                            |
| $\log \sin \theta_0 \left. \vphantom{\log \sin \theta_0} \right\}$                                  |                                            |
| $\theta_0 = 1^\circ 41' 31''.1$                                                                     |                                            |

$$\log \cos^2 \frac{1}{2}\theta = 9.987362$$

$$\log \cos^2 \frac{1}{2}\theta_0 = 9.999906$$

---


$$9.987268$$

$$\log \frac{\pi}{2} = 0.196120$$

---


$$\log K = 0.208852$$

$\theta_{00}$  is not calculated, as it is evident that its cosine will be 1.

EXAMPLE. Given  $k = \sin 75^\circ$ . Find  $K$ .

By eq. (7), Chap. V.

From eqs. (14<sub>1</sub>), Chap. IV, we find

$$k = \sin \theta = \sin 75^\circ \qquad \log = 9.9849438$$

$$k_0 = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta = \tan^2 37^\circ 30' \\ \sin \theta_0 = \sin 36^\circ 4' 16''.47 \end{array} \right\} \qquad 9.7699610$$

$$k_{00} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_0 = \tan^2 18^\circ 2' 8''.235 \\ \sin \theta_{00} = \sin 6^\circ 5' 9''.38 \end{array} \right\} \qquad 9.0253880$$

$$k_{03} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_{00} = \tan^2 3^\circ 2' 34''.69 \\ \sin \theta_{03} = \sin 9' 42''.90 \end{array} \right\} \qquad 7.4511672$$

|                                                       | log       | 2 log     | a. c. 2 log |
|-------------------------------------------------------|-----------|-----------|-------------|
| $\cos \frac{1}{2}\theta = \cos 37^\circ 30'$          | 9.8994667 | 9.7989334 | 0.2010666   |
| $\cos \frac{1}{2}\theta_0 = \cos 18^\circ 2'.13725$   | 9.9781184 | 9.9562368 | 0.0437632   |
| $\cos \frac{1}{2}\theta_{02} = \cos 3^\circ 2'.57817$ | 9.9993873 | 9.9987746 | 0.0012254   |
| $\cos \frac{1}{2}\theta_{03} = \cos 4'.8575$          | 9.9999995 | 9.9999990 | 0.0000010   |

---

0.2460562

$$\frac{\pi}{2}$$

$$\frac{\pi}{2}$$

.1961199

---


$$\log K = 0.4421761$$

$$K = 2.768064 \quad \text{Ans.}$$

EXAMPLE. Given  $k = \sin 45^\circ$ . Find  $K$ .

Method of eq. (7), Chap. V.

From eqs. (14<sub>1</sub>), Chap. IV, we have

$$k_0 = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta = \tan^2 22^\circ 30' \\ \sin \theta_0 = \sin 9^\circ 52'.75683 \end{array} \right\} \quad \begin{array}{l} \log \\ 9.2344486 \end{array}$$

$$k_{00} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_0 = \tan^2 4^\circ 56'.37841 \\ \sin \theta_{00} = \sin 25'.679 \end{array} \right\} \quad 7.8733009$$

$$k_{03} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_{00} = \tan^2 12'.3395 \\ \sin \theta_{03} = \sin 0'.05 \end{array} \right\} \quad 5.1445523$$

$$\text{a. c. log } \cos^2 \frac{1}{2}\theta \quad 0.0687694$$

$$\text{a. c. log } \cos^2 \frac{1}{2}\theta_0 \quad 0.0032320$$

$$\text{a. c. log } \cos^2 \frac{1}{2}\theta_{00} \quad 0.0000060$$

$$\log \frac{\pi}{2} \quad 0.1961199$$

---


$$\log K = 0.2681273$$

$$K = 1.8540747 \quad \text{Ans.}$$

EXAMPLE. Given  $\theta = 63^\circ 30'$ . Find  $K$ .

$$\text{Ans. } \log K = 0.3539686.$$

EXAMPLE. Given  $\theta = 34^\circ 30'$ . Find  $K$ .

$$\text{Ans. } K = 1.72627.$$

## CHAPTER XIV.

### NUMERICAL CALCULATIONS. *u*.

#### CALCULATION OF THE VALUE OF *u*.

WHEN  $\theta^\circ = \sin^{-1}k < 45^\circ$ .

EXAMPLE. Let  $\phi = 30^\circ$ ,  $k = \sin 45^\circ$ . Find *u*.

*First Method.* Eq. (23), Chap. IV, and eqs. (14<sub>1</sub>), (14<sub>2</sub>), (14<sub>3</sub>), Chap. IV.

By equations (14<sub>1</sub>),

$$\frac{\theta}{2} = 22^\circ 30';$$

$$\log \tan \frac{\theta}{2} = 9.6172243;$$

$$\log \tan^2 \frac{\theta}{2} = 9.2344486 = \log k_0 = \log \sin \theta_0;$$

$$\theta_0 = 9^\circ 52' 45''.41;$$

$$\log \tan \frac{\theta_0}{2} = 8.9366506;$$

$$\log \tan^2 \frac{\theta_0}{2} = 7.8733012 = \log k_{00} = \log \sin \theta_{00};$$

$$\theta_{00} = 0^\circ 25' 40''.7;$$

$$\log \tan^2 \frac{\theta_{00}}{2} = 5.144552 = \log k_{03}.$$

By equations (14),

$$\begin{aligned}\phi &= 30^\circ \\ \log \tan \phi &= 9.761439 \\ \log \cos \theta &= 9.849485 \\ \hline \log \tan (\phi_1 - \phi) &= 9.610924 \\ \phi_1 - \phi &= 22^\circ 12' 27''.56 \\ \phi_1 &= 52^\circ 12' 27''.56\end{aligned}$$

$$\begin{aligned}\log \tan \phi_1 &= 0.110438 \\ \log \cos \theta_0 &= 9.993512 \\ \hline \log \tan (\phi_2 - \phi_1) &= 0.103949 \\ \phi_2 - \phi_1 &= 51^\circ 47' 32''.59\end{aligned}$$

$$\begin{aligned}\phi_2 &= 104^\circ 0' 0''.15 \\ \log \tan \phi_2 &= 0.603228 \\ \log \cos \theta_{00} &= 9.999988 \\ \hline \log \tan (\phi_3 - \phi_2) &= 0.603216 \\ \phi_3 - \phi_2 &= 104^\circ 0' 1''.5\end{aligned}$$

$$\phi_3 = 208^\circ 0' 1''.65$$

$$\text{Since } \frac{\phi_2}{4} = 26^\circ 0' 0''.04 \quad \text{and} \quad \frac{\phi_3}{8} = 26^\circ 0' 0''.21,$$

we need not calculate  $\phi_4$ .

$$\frac{\phi_3}{8} = 93600''.21.$$

Reducing this to radians, we have

$$\log \frac{\phi_3}{8} = 9.656852.$$

Substituting in eq. (23), Chap. IV, we have, since  $\cos \theta_{03} = 1$ ,

$$\text{a. c. } \log \cos \theta = 0.150515$$

$$\log \cos \theta_0 = 9.993512$$

$$\log \cos \theta_{00} = 9.999988$$

$$\hline 0.144014$$

$$0.072007 = \log \sqrt{\frac{\cos \theta_0 \cos \theta_{00}}{\cos \theta}}$$

$$\log \frac{\phi_3}{8} = 9.656852$$

$$\log u = 9.728859$$

$$u = 0.535623, \quad \text{Ans.}$$

When  $\theta = \sin^{-1} k > 45^\circ$ .

EXAMPLE. Given  $k = \sin 75^\circ$ ,  $\tan \phi = \sqrt{\frac{2}{\sqrt{3}}}$ . To find  $F(k, \phi)$ .

*First Method. Bisected Amplitudes.*

By equations (24) and (25), Chap. IV, we get

$$\phi = 47^\circ 3' 30''.91,$$

$$\phi_{\frac{3}{2}} = 25^\circ 36' 5''.64,$$

$$\phi_{\frac{4}{3}} = 13^\circ 6' 30''.98,$$

$$\phi_{\frac{5}{3}} = 6^\circ 35' 40''.74,$$

$$\phi_{\frac{15}{8}} = 3^\circ 18' 8''.75,$$

$$\phi_{\frac{37}{8}} = 1^\circ 39' 7''.43,$$

$$\beta = 45^\circ;$$

$$\beta_0 = 24^\circ 40' 10''.94;$$

$$\beta_{00} = 12^\circ 39' 15''.83;$$

$$\beta_{03} = 6^\circ 22' 8''.40;$$

$$\beta_{04} =$$

Substituting in equation (26), Chap. IV, we have

$$F(k, \phi) = 32 \times 1^\circ 39' 7''.43$$

$$= 52^\circ 51' 58''.03$$

$$= 0.9226878. \quad \text{Ans.}$$

*Second Method.* Equation (29), Chap. IV.

From equations (18<sub>3</sub>), Chap. IV, we have

|                                                                                                                                                              |                              | log       |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------|-----------|
| $k = \cos \eta$                                                                                                                                              | $= \cos 15^\circ 0' 0''.00$  | 9.9849438 |
| $k' = \sin \eta$                                                                                                                                             | $= \sin 15^\circ 0' 0''.00$  | 9.4129962 |
| $k'_0 = \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \eta = \tan^2 7^\circ 30' 0''.00 \\ \sin \eta_0 = \sin 0^\circ 59' 35''.25 \end{array} \right\}$         |                              | 8.2388582 |
| $k_1 = \cos \eta_0$                                                                                                                                          | $= \cos 0^\circ 59' 35''.25$ | 9.9999348 |
| $k'_{00} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \eta_0 = \tan^2 0^\circ 29' 47''.62 \\ \sin \eta_{00} = \sin 0^\circ 0' 15''.49 \end{array} \right\}$ |                              | 5.8757219 |
| $k_2 = \cos \eta_{00}$                                                                                                                                       | $= \cos 0^\circ 0' 15''.49$  | 0.0000000 |
| $k'_{03} = (\frac{1}{2} k'_{00})^2$                                                                                                                          |                              | 1.1493838 |

From equations (18<sub>2</sub>), Chap. IV, we get

$$\begin{aligned}
 \phi &= 47^\circ 3' 30''.95; \\
 2\phi_0 - \phi &= 45^\circ; \\
 \phi_0 &= 46^\circ 1' 45''.475; \\
 \phi_{02} &= 46^\circ 1' 29''.41; \\
 \phi_{03} &= 46^\circ 1' 29''.41; \\
 45^\circ + \frac{1}{2} \phi_3 &= 68^\circ 0' 44''.705.
 \end{aligned}$$

Substituting these values in eq. (29), Chap. IV, we get

$$\begin{aligned}
 F(k, \phi) &= \sqrt{\frac{k_1}{k}} \cdot \frac{1}{M} \cdot \log \tan 68^\circ 0' 44''.705 \\
 &= 0.9226877. \quad \text{Ans.}
 \end{aligned}$$

*Third Method.* Equation (23)\*, Chap. IV.

From equations (14<sub>1</sub>), Chap. IV, we have

$$\begin{aligned}
 k &= \sin \theta &= \sin 75^\circ & \text{o' o''} & \log = 9.9849438 \\
 k' &= \cos \theta &= \cos 75^\circ & & 9.4129962 \\
 k_0 &= \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \theta = \tan^2 37^\circ 30' \\ \sin \theta_0 = \sin 36^\circ 4' 16''.47 \end{array} \right\} & & & 9.7699610 \\
 k_1' &= \cos \theta_0 & & & 9.9075648 \\
 k_{02} &= \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \theta_0 = \tan^2 18^\circ 2' 8''.235 \\ \sin \theta_{00} = \sin 6^\circ 5' 9''.38 \end{array} \right\} & & & 9.0253880 \\
 k_2' &= \cos \theta_{00} & & & 9.9975452 \\
 k_{03} &= \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \theta_{00} = \tan^2 3^\circ 2' 34''.69 \\ \sin \theta_{03} = \sin 9' 42''.90 \end{array} \right\} & & & 7.4511672 \\
 k_3' &= \cos \theta_{03} & & & 9.9999982 \\
 k_{04} &= \left( \frac{1}{2} k_{03} \right)^2 & & & 4.3002761 \\
 k_4' &= & & & 0.0000000
 \end{aligned}$$

From equations (14<sub>2</sub>), Chap. IV, we have

$$\begin{aligned}
 \phi &= 47^\circ 3' 30''.94; \\
 \phi_1 &= 62^\circ 36' 3''.10; \\
 \phi_2 &= 119^\circ 55' 47''.67; \\
 \phi_3 &= 240^\circ \text{ o' o''}.19; \\
 \phi_4 &= 480^\circ \text{ o' o''}.
 \end{aligned}$$

Therefore the limit of  $\phi$ ,  $\frac{\phi_1}{2}$ ,  $\frac{\phi_2}{4}$ , or  $\frac{\phi_n}{2^n}$  is  $30^\circ = \frac{\pi}{6}$ .

Substituting these values in eq. (23)\*, Chap. IV, we have

$$\begin{aligned}
 F(k, \phi) &= \sqrt{\frac{k_1' k_2' k_3' k_4'}{k'}} \cdot \frac{\pi}{6} \\
 &= 0.9226874. \quad \text{Ans.}
 \end{aligned}$$

EXAMPLE. Given  $\phi = 30^\circ$ ,  $k = \sin 89^\circ$ . Find  $u$ .

Method of eq. (28), Chap. IV.

From eqs. (18<sub>1</sub>) we find

$$k_1 = \sin \theta_1 \quad \text{and} \quad \tan^2 \frac{1}{2} \theta_1 = k = \sin \theta,$$

from which we find that  $k_1 = 1$  as far as seven decimal places.

From eqs. (18<sub>2</sub>) we have

$$\begin{aligned} \sin \phi &= 9.6989700 \\ k &= 9.9999338 \\ \sin (2\phi_0 - \phi) &= 9.6989038 \\ 2\phi_0 - \phi &= 29^\circ 59' .69733 \\ 2\phi_0 &= 59^\circ 59' .69733 \\ 45^\circ + \frac{1}{2} \phi_0^* &= 59^\circ 59' .92433 \\ \log (45^\circ + \frac{1}{2} \phi_0) &= 0.2385385 \end{aligned}$$

From eqs. (18<sub>3</sub>), Chap. IV, we have

$$k = \cos \eta = \cos 1^\circ, \quad \frac{1}{2} \eta = 30'.$$

Substituting in eq. (28), Chap. IV, we have

$$\begin{aligned} \text{a. c. } \log \cos \frac{1}{2} \eta &= 0.0000330 \\ \log \log (45^\circ + \frac{1}{2} \phi_0) &= 9.3775585 \\ \text{a. c. } \log M &= 0.3622157 \\ \log F(k, \phi) &= 9.7398072 \\ F(k, \phi) &= 0.549297. \quad \text{Ans.} \end{aligned}$$

EXAMPLE. Given  $\phi = 79^\circ$ ,  $k = 0.25882$ . Find  $u$ .  
*Ans.  $u = 0.39947$ .*

EXAMPLE. Given  $\phi = 37^\circ$ ,  $k = 0.86603$ . Find  $u$ .  
*Ans.  $u = 0.68141$ .*

\* Since  $k_1 = 1$ ,  $\phi_{00} = \phi_0$ , and we need not carry the calculation further.

## CHAPTER XV.

### NUMERICAL CALCULATIONS. $\phi$ .

EXAMPLE. Given  $u = 1.368407$ ,  $\theta = 38^\circ$ . Find  $\phi$ .

*First Method.* From eqs. (46) and (41)\*, Chap. IX, we have

$$u = x\Theta^2(K),$$

$$\Delta\phi = \sqrt{k'} \frac{\Theta_1(x)}{\Theta(x)}.$$

From equations (5), Chap. XII, and (38), Chap. IX, we have

$$\begin{aligned} \log q &= 8.4734187 \\ \log \Theta^2(K) &= 0.0501955 \\ \log u &= 0.1362153 \\ \hline \log x &= 0.0860198 \\ x &= 69^\circ 50' 46''.12 \end{aligned}$$

From equations (23) and (24), Chap. IX, we get

$$\begin{aligned} \log \Theta_1(x) &= 9.9798368 \\ \log \Theta(x) &= 0.0192687 \\ \hline &9.9605681 \\ \log \sqrt{k'} &= 9.9482661 \\ \hline \log \Delta\phi &= 9.9088342 = \log \sin \lambda \end{aligned}$$

But

$$\begin{aligned} k^2 \sin^2 \phi &= 1 - \Delta^2 \phi, \\ k \sin \phi &= \cos \lambda; \end{aligned}$$

whence

$$\log \cos \lambda = 9.7675483$$

$$\log k = 9.7893420$$

$$\log \sin \phi = 9.9782063$$

$$\phi = 72^\circ. \text{ Ans.}$$

*Second Method.* From eq. (1), Chap. VI.

From eqs. (14<sub>1</sub>) Chap. IV, we find

$$k_0 = \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \theta = \tan^2 19^\circ \\ \sin \theta_0 = \sin 6^\circ 48' .54569 \\ \cos \theta_0 \end{array} \right\} \log = 9.0739438$$

$$9.9969260$$

$$k_{00} = \left\{ \begin{array}{l} \tan^2 \frac{1}{3} \theta_0 = \tan^2 3^\circ 24' .2784 \\ \sin \theta_{00} = \sin 12' .16659 \\ \cos \theta_{00} \end{array} \right\} \log = 7.5488952$$

$$9.9999974$$

$$k_{03} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2} \theta_{03} = \tan^2 6' .08329 \\ \sin \theta_{03} \\ \cos \theta_{03} \end{array} \right\} \log = 4.4957316$$

$$0.0000000$$

Substituting these values in eq. (1), Chap. VI, we have

$$\log \cos \theta_0 \quad 9.9969260$$

$$\log \cos \theta_{00} \quad 9.9999974$$

$$9.9969234$$

$$\log \sqrt{\cos \theta_0 \cos \theta_{00}} \quad 9.9984617$$

$$\text{a. c. log} \quad \text{“} \quad \text{“} \quad 0.0015383$$

$$\log u \quad .1362153$$

$$\log \sqrt{\cos \theta} \quad 9.9482660$$

$$\log 2^3 \quad .9030900^*$$

$$\text{a. c. log} \sqrt{\cos \theta_0 \cos \theta_{00}} \quad 0.0015383$$

$$0.9891096$$

$$2.2418773$$

$$\log \phi_s^* \quad 2.7472323$$

$$\phi_s \quad 558^\circ 46' .140$$

---

\*  $n$  is taken equal to 3, because  $\cos_{03} = 1$ .

Whence, by equations (1)\* of Chap. VI, we get

$$\begin{array}{rcl}
 k_{03} \log & = & 4.4957316 \\
 \sin \phi_3 & & 9.5075232_n \\
 \hline
 \sin (2\phi_2 - \phi_3) & & 4.0032548_n \\
 2\phi_2 - \phi_3 & = & -0'.00346 \\
 \phi_2 & = & 279^\circ 23'.06827 \\
 \\
 k_{00} \log & = & 7.5488952 \\
 \sin \phi_2 & & 9.9941484_n \\
 \hline
 \sin (2\phi_1 - \phi_2) & & 7.5430436_n \\
 2\phi_1 - \phi_2 & = & -12'.0039 \\
 \phi_1 & = & 139^\circ 35'.5321 \\
 \\
 k_0 \log & = & 9.0739438 \\
 \sin \phi_1 & & 9.8117249 \\
 \hline
 \sin (2\phi - \phi_1) & & 8.8856687 \\
 2\phi - \phi_1 & = & 4^\circ 24'.467 \\
 \phi & = & 71^\circ 59'.9999 \\
 & = & 72^\circ. \text{ Ans.}
 \end{array}$$

EXAMPLE. Given  $u = 2.41569$ ,  $\theta = 80^\circ$ . Find  $\phi$ .

*Ans.*  $\phi = 82^\circ$ .

EXAMPLE. Given  $u = 1.62530$ ,  $k = \frac{1}{2}$ . Find  $\phi$ .

*Ans.*  $\phi = 87^\circ$ .

## CHAPTER XVI.

### NUMERICAL CALCULATIONS. $E(k, \phi)$ .

*First Method.* By Chap. X, eqs. (15), (16), and (17).

EXAMPLE. Given  $k = 0.9327$ ,  $\phi = 80^\circ$ . Find  $E(k, \phi)$ .

By eq. (15), Chap. X,

$$\begin{array}{ll}
 \phi = 80^\circ; & \gamma = 67^\circ 44'.; \\
 \phi_{\frac{1}{2}} = 50^\circ 43'.6, & \gamma_{\frac{1}{2}} = 46^\circ 40'.4; \\
 \phi_{\frac{1}{4}} = 27^\circ 48'.5, & \gamma_{\frac{1}{4}} = 26^\circ 0'.1; \\
 \phi_{\frac{1}{8}} = 14^\circ 16'.7, & \gamma_{\frac{1}{8}} = 13^\circ 24'.0; \\
 \phi_{\frac{1}{16}} = 7^\circ 11'.3, & \gamma_{\frac{1}{16}} = 6^\circ 45'.2; \\
 \phi_{\frac{1}{32}} = 3^\circ 36'.0, & \log \sin \gamma_{\frac{1}{32}} = 8.77094; \\
 \phi_{\frac{1}{32}} = 0.062831. & \\
 \therefore \phi_{\frac{1}{32}}^6 < 0.0000001. &
 \end{array}$$

Whence, by eq. (17),

$$\begin{array}{rcl}
 E(k, \phi_{\frac{1}{32}}) & = & 0.062794 \\
 \sin \phi & \sin^2 \gamma_{\frac{1}{2}} & = 0.52116 \\
 2 \sin \phi_{\frac{1}{2}} & \sin^2 \gamma_{\frac{1}{4}} & = 0.29757 \\
 4 \sin \phi_{\frac{1}{4}} & \sin^2 \gamma_{\frac{1}{8}} & = 0.10023 \\
 8 \sin \phi_{\frac{1}{8}} & \sin^2 \gamma_{\frac{1}{16}} & = 0.02728 \\
 16 \sin \phi_{\frac{1}{16}} & \sin^2 \gamma_{\frac{1}{32}} & = 0.00697 \\
 & & \hline
 & & 0.95321
 \end{array}$$

Hence, by eq. (16),

$$\begin{aligned}
 E(k, \phi) &= 32E(k, \phi_{\frac{1}{32}}) - 0.95321 \\
 &= 2.0094 - 0.9532 = 1.0562.
 \end{aligned}$$

*Second Method.* By Chap. X, eq. (26).

EXAMPLE. Given  $k = \sin 75^\circ$ ,  $\tan \phi = \sqrt{\frac{2}{\sqrt{3}}}$ . Find  $E(k, \phi)$ .

From eqs. (14<sub>1</sub>), Chap. IV, we have

$$k = \sin \theta = \sin 75^\circ 0' 0'' \quad \log = 9.9849438$$

$$k' = \cos \theta = \cos 75^\circ \quad \quad \quad 9.4129962$$

$$k_0 = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta = \tan^2 37^\circ 30' \\ \sin \theta_0 = \sin 36^\circ 4' 16''.47 \end{array} \right\} \quad 9.7699610$$

$$k_1' = \cos \theta_0 \quad \quad \quad 9.9075648$$

$$k_{02} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_0 = \tan^2 18^\circ 2' 8''.235 \\ \sin \theta_{00} = \sin 6^\circ 5' 9''.38 \end{array} \right\} \quad 9.0253880$$

$$k_2' = \cos \theta_{00} \quad \quad \quad 9.9975452$$

$$k_{03} = \left\{ \begin{array}{l} \tan^2 \frac{1}{2}\theta_{00} = \tan^2 3^\circ 2' 34''.69 \\ \sin \theta_{03} = \sin 9' 42''.90 \end{array} \right\} \quad 7.4511672$$

$$k_3' = \cos \theta_{03} \quad \quad \quad 9.9999982$$

$$k_{04} = (\frac{1}{2}k_{03})^2 \quad \quad \quad 4.3002761$$

$$k_4' = \quad \quad \quad 0.0000000$$

From eqs. (14<sub>2</sub>), Chap. IV, we have

$$\phi = 47^\circ 3' 30''.94;$$

$$\phi_1 = 62^\circ 36' 3''.10;$$

$$\phi_2 = 119^\circ 55' 47''.67;$$

$$\phi_3 = 240^\circ 0' 0''.19.$$

Applying eq. (26), Chap. X, we have

|          |                    |                 |
|----------|--------------------|-----------------|
| $k^2$    | $\log = 9.9698876$ |                 |
| a. c. 2  | <u>9.6989700</u>   |                 |
|          | 9.6688576          | .4665064        |
| $k_0$    | 9.7699610          |                 |
| a. c. 2  | <u>9.6989700</u>   |                 |
|          | 9.1377886          | .1373373        |
| $k_{00}$ | 9.0253880          |                 |
| a. c. 2  | <u>9.6989700</u>   |                 |
|          | 7.8621466          | .0072802        |
| $k_{03}$ | 7.4511672          |                 |
| a. c. 2  | <u>9.6989700</u>   |                 |
|          | 5.0132838          | .0000103        |
|          |                    | <u>.6111342</u> |

$$1 - .6111342 = 0.3888658.$$

From eq. (23)\*, Chap. IV, we find  $F(k, \phi) = 0.9226874$ .  
Hence

$$F(k, \phi) \left[ 1 - \frac{k^2}{2} \left( 1 + \frac{k_0}{2} + \dots \right) \right] = 0.3588016$$

$$\frac{k \sqrt{k_0}}{2} \sin \phi_1 = 0.3290186$$

$$\frac{k \sqrt{k_0 k_{00}}}{4} \sin \phi_2 = 0.0522872$$

$$\frac{k \sqrt{k_0 k_{02} k_{03}}}{8} \sin \phi_3 = -0.0013888$$

$$\frac{k \sqrt{k_0 \dots k_{04}}}{16} \sin \phi_4 = 0.0000010$$


---


$$0.3799180$$

Whence

$$E(k, \phi) = 0.3588016 + 0.3799180 = 0.7387196. \quad \text{Ans.}$$

EXAMPLE. Given  $k = \sin 75^\circ$ . Find  $E\left(k, \frac{\pi}{2}\right)$ .

From Example 2, Chap. XIII, we find

$$\begin{aligned} \log F\left(k, \frac{\pi}{2}\right) &= 0.4421761 \\ \log 0.3888658 &= \underline{1.5897998} \\ \log E\left(k, \frac{\pi}{2}\right) &= 0.0319759 \end{aligned}$$

$$E\left(k, \frac{\pi}{2}\right) = 1.076405. \quad \text{Ans.}$$

EXAMPLE. Given  $k = \sin 30^\circ$ ,  $\phi = 81^\circ$ . Find  $E(k, \phi)$ .

$$\text{Ans. } E(k, \phi) = 1.33124.$$

EXAMPLE. Find  $E(\sin 80^\circ, 55^\circ)$ . Ans. 0.82417.

EXAMPLE. Find  $E\left(\sin 27^\circ, \frac{\pi}{2}\right)$ . Ans. 1.48642.

EXAMPLE. Find  $E(\sin 19^\circ, 27^\circ)$ . Ans. 0.46946.

given point of the hyperbola on this line with the centre. The angle which this joining line makes with the axis of  $x$  we will call  $\phi$ . If  $y$  is the ordinate of the point on the hyperbola, then evidently

$$y = \frac{b^2 \tan \phi}{\sqrt{a^2 + b^2}},$$

and

$$x = \frac{a}{\cos \phi} \sqrt{1 - \frac{a^2 \sin^2 \phi}{a^2 + b^2}} = \frac{a}{\cos \phi} \sqrt{1 - \frac{1}{e^2} \sin^2 \phi};$$

whence

$$\begin{aligned} s &= \int \sqrt{ax^2 + dy^2} = \frac{b^2}{c} \int_0^\phi \frac{d\phi}{\cos^2 \phi \sqrt{1 - \frac{1}{e^2} \sin^2 \phi}} \\ &= \frac{b^2}{c} \int_0^\phi \frac{d\phi}{\cos^2 \phi \sqrt{1 - k^2 \sin^2 \phi}}. \end{aligned}$$

But

$$\begin{aligned} d(\tan \phi \sqrt{1 - k^2 \sin^2 \phi}) &= d\phi \sqrt{1 - k^2 \sin^2 \phi} + d\phi \frac{1 - k^2}{\sqrt{1 - e^2 \sin^2 \phi}} \\ &\quad - \frac{1 - k^2}{\cos^2 \phi \sqrt{1 - e^2 \sin^2 \phi}} d\phi. \end{aligned}$$

Consequently

$$\begin{aligned} s &= \frac{b^2}{c} \int_0^\phi \frac{d\phi}{\cos^2 \phi \sqrt{1 - k^2 \sin^2 \phi}} \\ &= \frac{b^2}{c} F(k, \phi) - cE(k, \phi) + c \tan \phi \Delta(k, \phi) \\ &= \frac{b^2}{ae} F\left(\frac{1}{e}, \phi\right) - aeE\left(\frac{1}{e}, \phi\right) + ae \tan \phi \Delta\left(\frac{1}{e}, \phi\right). \end{aligned}$$

EXAMPLE. Find the length of the arc of the hyperbola  $\frac{x^2}{20.25} - \frac{y^2}{400} = 1$  from the vertex to the point whose ordinate is  $\frac{40}{2.05} \tan 15^\circ$ . *Ans.* 5.231184.

EXAMPLE. Find the length of the arc of the hyperbola  $\frac{x^2}{144} - \frac{y^2}{81} = 100$  from the vertex to the point whose ordinate is 0.6. *Ans.* 0.6582.

# SHORT-TITLE CATALOGUE

OF THE

## PUBLICATIONS

OF

## JOHN WILEY & SONS

NEW YORK

LONDON: CHAPMAN & HALL, LIMITED

---

ARRANGED UNDER SUBJECTS

---

Descriptive circulars sent on application. Books marked with an asterisk (\*) are sold at *net* prices only. All books are bound in cloth unless otherwise stated.

---

### AGRICULTURE—HORTICULTURE—FORESTRY.

|                                                                                 |       |        |
|---------------------------------------------------------------------------------|-------|--------|
| Armsby's Principles of Animal Nutrition.....                                    | 8vo,  | \$4 00 |
| Budd and Hansen's American Horticultural Manual:                                |       |        |
| Part I. Propagation, Culture, and Improvement.....                              | 12mo, | 1 50   |
| Part II. Systematic Pomology.....                                               | 12mo, | 1 50   |
| Elliott's Engineering for Land Drainage.....                                    | 12mo, | 1 50   |
| Practical Farm Drainage. (Second Edition, Rewritten.).....                      | 12mo, | 1 50   |
| Graves's Forest Mensuration.....                                                | 8vo,  | 4 00   |
| Green's Principles of American Forestry.....                                    | 12mo, | 1 50   |
| Grotenfelt's Principles of Modern Dairy Practice. (Woll.).....                  | 12mo, | 2 00   |
| * Herrick's Denatured or Industrial Alcohol.....                                | 8vo,  | 4 00   |
| Kemp and Waugh's Landscape Gardening. (New Edition, Rewritten. In Preparation.) |       |        |
| * McKay and Larsen's Principles and Practice of Butter-making.....              | 8vo,  | 1 50   |
| Maynard's Landscape Gardening as Applied to Home Decoration.....                | 12mo, | 1 50   |
| Sanderson's Insects Injurious to Staple Crops.....                              | 12mo, | 1 50   |
| Sanderson and Headlee's Insects Injurious to Garden Crops. (In Preparation.)    |       |        |
| * Schwarz's Longleaf Pine in Virgin Forests.....                                | 12mo, | 1 25   |
| Stockbridge's Rocks and Soils.....                                              | 8vo,  | 2 50   |
| Winton's Microscopy of Vegetable Foods.....                                     | 8vo,  | 7 50   |
| Woll's Handbook for Farmers and Dairymen.....                                   | 16mo, | 1 50   |

### ARCHITECTURE.

|                                                            |       |      |
|------------------------------------------------------------|-------|------|
| Baldwin's Steam Heating for Buildings.....                 | 12mo, | 2 50 |
| Berg's Buildings and Structures of American Railroads..... | 4to,  | 5 00 |
| Birkmire's Architectural Iron and Steel.....               | 8vo,  | 3 50 |
| Compound Riveted Girders as Applied in Buildings.....      | 8vo,  | 2 00 |
| Planning and Construction of American Theatres.....        | 8vo,  | 3 00 |
| Planning and Construction of High Office Buildings.....    | 8vo,  | 3 50 |
| Skeleton Construction in Buildings.....                    | 8vo,  | 3 00 |

|                                                                                                                                                      |             |        |
|------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|--------|
| Briggs's Modern American School Buildings.....                                                                                                       | 8vo,        | \$4 00 |
| Byrne's Inspection of Materials and Workmanship Employed in Construction.                                                                            | 16mo,       | 3 00   |
| Carpenter's Heating and Ventilating of Buildings.....                                                                                                | 8vo,        | 4 00   |
| * Corthell's Allowable Pressure on Deep Foundations.....                                                                                             | 12mo,       | 1 25   |
| Freitag's Architectural Engineering.....                                                                                                             | 8vo,        | 3 50   |
| Fireproofing of Steel Buildings.....                                                                                                                 | 8vo,        | 2 50   |
| Gerhard's Guide to Sanitary Inspections. (Fourth Edition, Entirely Revised and Enlarged.).....                                                       | 12mo,       | 1 50   |
| * Modern Baths and Bath Houses.....                                                                                                                  | 8vo,        | 3 00   |
| Sanitation of Public Buildings.....                                                                                                                  | 12mo,       | 1 50   |
| Theatre Fires and Panics.....                                                                                                                        | 12mo,       | 1 50   |
| * The Water Supply, Sewerage and Plumbing of Modern City Buildings.                                                                                  | 8vo,        | 4 00   |
| Johnson's Statics by Algebraic and Graphic Methods.....                                                                                              | 8vo,        | 2 00   |
| Kellaway's How to Lay Out Suburban Home Grounds.....                                                                                                 | 8vo,        | 2 00   |
| Kidder's Architects' and Builders' Pocket-book.....                                                                                                  | 16mo, mor., | 5 00   |
| Merrill's Stones for Building and Decoration.....                                                                                                    | 8vo,        | 5 00   |
| Monckton's Stair building.....                                                                                                                       | 4to,        | 4 00   |
| Patton's Practical Treatise on Foundations.....                                                                                                      | 8vo,        | 5 00   |
| Peabody's Naval Architecture.....                                                                                                                    | 8vc,        | 7 50   |
| Rice's Concrete-block Manufacture.....                                                                                                               | 8vo,        | 2 00   |
| Richey's Handbook for Superintendents of Construction.....                                                                                           | 16mo, mor., | 4 00   |
| Building Foreman's Pocket Book and Ready Reference.....                                                                                              | 16mo, mor., | 5 00   |
| * Building Mechanics' Ready Reference Series:                                                                                                        |             |        |
| * Carpenters' and Woodworkers' Edition.....                                                                                                          | 16mo, mor., | 1 50   |
| * Cement Workers' and Plasterers' Edition.....                                                                                                       | 16mo, mor., | 1 50   |
| * Plumbers', Steam-Fitters', and Tanners' Edition.....                                                                                               | 16mo, mor., | 1 50   |
| * Stone- and Brick-masons' Edition.....                                                                                                              | 16mo, mor., | 1 50   |
| Sabin's House Painting.....                                                                                                                          | 12mo,       | 1 00   |
| Siebert and Biggin's Modern Stone-cutting and Masonry.....                                                                                           | 8vo,        | 1 50   |
| Snow's Principal Species of Wood.....                                                                                                                | 8vo,        | 3 50   |
| Towne's Locks and Builders' Hardware.....                                                                                                            | 18mo, mor., | 3 00   |
| Wait's Engineering and Architectural Jurisprudence.....                                                                                              | 8vo,        | 6 00   |
| Sheep,                                                                                                                                               |             | 6 50   |
| Law of Contracts.....                                                                                                                                | 8vo,        | 3 00   |
| Law of Operations Preliminary to Construction in Engineering and Architecture.....                                                                   | 8vo,        | 5 00   |
| Sheep,                                                                                                                                               |             | 5 50   |
| Wilson's Air Conditioning.....                                                                                                                       | 12mo,       | 1 50   |
| Worcester and Atkinson's Small Hospitals, Establishment and Maintenance, Suggestions for Hospital Architecture, with Plans for a Small Hospital..... | 12mo,       | 1 25   |

## ARMY AND NAVY.

|                                                                                             |             |      |
|---------------------------------------------------------------------------------------------|-------------|------|
| Bernadou's Smokeless Powder, Nitro-cellulose, and the Theory of the Cellulose Molecule..... | 12mo,       | 2 50 |
| Chase's Art of Pattern Making.....                                                          | 12mo,       | 2 50 |
| Screw Propellers and Marine Propulsion.....                                                 | 8vo,        | 3 00 |
| * Cloke's Enlisted Specialists' Examiner.....                                               | 8vo,        | 2 00 |
| Gunner's Examiner.....                                                                      | 8vo,        | 1 50 |
| Craig's Azimuth.....                                                                        | 4to,        | 3 50 |
| Crehore and Squier's Polarizing Photo-chronograph.....                                      | 8vo,        | 3 50 |
| * Davis's Elements of Law.....                                                              | 8vo,        | 2 50 |
| * Treatise on the Military Law of United States.....                                        | 8vo,        | 7 00 |
| DeBrack's Cavalry Outpost Duties. (Carr.).....                                              | 24mo, mor., | 2 00 |
| * Dudley's Military Law and the Procedure of Courts-martial.....                            | Large 12mo, | 2 50 |
| Durand's Resistance and Propulsion of Ships.....                                            | 8vo,        | 5 00 |
| * Dyer's Handbook of Light Artillery.....                                                   | 12mo,       | 3 00 |
| Eissler's Modern High Explosives.....                                                       | 8vo,        | 4 00 |
| * Fieberger's Text-book on Field Fortification.....                                         | Large 12mo, | 2 00 |
| Hamilton and Bond's The Gunner's Catechism.....                                             | 18mo,       | 1 00 |
| * Hoff's Elementary Naval Tactics.....                                                      | 8vo,        | 1 50 |

|                                                                           |                |        |
|---------------------------------------------------------------------------|----------------|--------|
| Ingalls's Handbook of Problems in Direct Fire.....                        | 8vo,           | \$4 00 |
| * Lissak's Ordnance and Gunnery.....                                      | 8vo,           | 6 00   |
| * Ludlow's Logarithmic and Trigonometric Tables.....                      | 8vo,           | 1 00   |
| * Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II.....     | 8vo, each,     | 6 00   |
| * Mahan's Permanent Fortifications. (Mercur.).....                        | 8vo, half mor, | 7 50   |
| Manual for Courts-martial.....                                            | 16mo, mor,     | 1 50   |
| * Mercur's Attack of Fortified Places.....                                | 12mo,          | 2 00   |
| * Elements of the Art of War.....                                         | 8vo,           | 4 00   |
| Nixon's Adjutants' Manual.....                                            | 24mo,          | 1 00   |
| Peabody's Naval Architecture.....                                         | 8vo,           | 7 50   |
| * Phelps's Practical Marine Surveying.....                                | 8vo,           | 2 50   |
| Putnam's Nautical Charts.....                                             | 8vo,           | 2 00   |
| Rust's Ex-meridian Altitude, Azimuth and Star-Finding Tables.....         | 8vo            | 5 00   |
| Sharpe's Art of Subsisting Armies in War.....                             | 18mo, mor      | 1 50   |
| * Tupes and Poole's Manual of Bayonet Exercises and Musketry Fencing..... | 24mo, leather, | 50     |
| * Weaver's Military Explosives.....                                       | 8vo,           | 3 00   |
| * Woodhuil's Military Hygiene for Officers of the Line.....               | Large 12mo,    | 1 50   |

## ASSAYING.

|                                                                                               |            |      |
|-----------------------------------------------------------------------------------------------|------------|------|
| Betts's Lead Refining by Electrolysis.....                                                    | 8vo,       | 4 00 |
| Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe.....             | 16mo, mor, | 1 50 |
| Furman and Pardoe's Manual of Practical Assaying. (Sixth Edition, Revised and Enlarged.)..... | 8vo,       | 3 00 |
| Lodge's Notes on Assaying and Metallurgical Laboratory Experiments.....                       | 8vo,       | 3 00 |
| Low's Technical Methods of Ore Analysis.....                                                  | 8vo,       | 3 00 |
| Miller's Cyanide Process.....                                                                 | 12mo,      | 1 00 |
| Manual of Assaying.....                                                                       | 12mo,      | 1 00 |
| Minet's Production of Aluminum and its Industrial Use. (Waldo.).....                          | 12mo,      | 2 50 |
| O'Driscoll's Notes on the Treatment of Gold Ores.....                                         | 8vo,       | 2 00 |
| Ricketts and Miller's Notes on Assaying.....                                                  | 8vo,       | 3 00 |
| Robine and Lenglen's Cyanide Industry. (Le Clerc.).....                                       | 8vo,       | 4 00 |
| Seamon's Manual for Assayers and Chemists. (In Press.).....                                   |            |      |
| Ulke's Modern Electrolytic Copper Refining.....                                               | 8vo,       | 3 00 |
| Wilson's Chlorination Process.....                                                            | 12mo,      | 1 50 |
| Cyanide Processes.....                                                                        | 12mo,      | 1 50 |

## ASTRONOMY.

|                                                                   |            |      |
|-------------------------------------------------------------------|------------|------|
| Comstock's Field Astronomy for Engineers.....                     | 8vo,       | 2 50 |
| Craig's Azimuth.....                                              | 4to,       | 3 50 |
| Crandall's Text-book on Geodesy and Least Squares.....            | 8vo,       | 3 00 |
| Doolittle's Treatise on Practical Astronomy.....                  | 8vo,       | 4 00 |
| Hayford's Text-book of Geodetic Astronomy.....                    | 8vo,       | 3 00 |
| Hosmer's Azimuth.....                                             | 16mo, mor, | 1 00 |
| Merriman's Elements of Precise Surveying and Geodesy.....         | 8vo,       | 2 50 |
| * Michie and Harlow's Practical Astronomy.....                    | 8vo,       | 3 00 |
| Rust's Ex-meridian Altitude, Azimuth and Star-Finding Tables..... | 8vo,       | 5 00 |
| * White's Elements of Theoretical and Descriptive Astronomy.....  | 12mo,      | 2 00 |

## CHEMISTRY.

|                                                                                     |             |      |
|-------------------------------------------------------------------------------------|-------------|------|
| * Abderhalden's Physiological Chemistry in Thirty Lectures. (Hall and Defren.)..... | 8vo,        | 5 00 |
| * Abegg's Theory of Electrolytic Dissociation. (von Ende.).....                     | 12mo,       | 1 25 |
| Alexeyeff's General Principles of Organic Syntheses. (Matthews.).....               | 8vo,        | 3 00 |
| Allen's Tables for Iron Analysis.....                                               | 8vo,        | 3 00 |
| Armsby's Principles of Animal Nutrition.....                                        | 8vo,        | 4 00 |
| Arnold's Compendium of Chemistry. (Mandel.).....                                    | Large 12mo, | 3 50 |

|                                                                                                                  |              |        |
|------------------------------------------------------------------------------------------------------------------|--------------|--------|
| Association of State and National Food and Dairy Departments, Hartford Meeting, 1906.....                        | 8vo,         | \$3 00 |
| Jamestown Meeting, 1907.....                                                                                     | 8vo,         | 3 00   |
| Austen's Notes for Chemical Students.....                                                                        | 12mo,        | 1 50   |
| Baskerville's Chemical Elements. (In Preparation.)                                                               |              |        |
| Bernadou's Smokeless Powder.—Nitro-cellulose, and Theory of the Cellulose Molecule.....                          | 12mo,        | 2 50   |
| * Biltz's Introduction to Inorganic Chemistry. (Hall and Phelan.)                                                | 12mo,        | 1 25   |
| Laboratory Methods of Inorganic Chemistry. (Hall and Blanchard.)                                                 | 8vo,         | 3 00   |
| * Blanchard's Synthetic Inorganic Chemistry.....                                                                 | 12mo,        | 1 00   |
| * Browning's Introduction to the Rarer Elements.....                                                             | 8vo,         | 1 50   |
| * Claassen's Beet-sugar Manufacture. (Hall and Rolfe.).....                                                      | 8vo,         | 3 00   |
| Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.)                                            | 8vo,         | 3 00   |
| Cohn's Indicators and Test-papers.....                                                                           | 12mo,        | 2 00   |
| Tests and Reagents.....                                                                                          | 8vo,         | 3 00   |
| * Danneel's Electrochemistry. (Merriam.).....                                                                    | 12mo,        | 1 25   |
| Dannerth's Methods of Textile Chemistry.....                                                                     | 12mo,        | 2 00   |
| Duhem's Thermodynamics and Chemistry. (Burgess.).....                                                            | 8vo,         | 4 00   |
| Effront's Enzymes and their Applications. (Prescott.).....                                                       | 8vo,         | 3 00   |
| Eissler's Modern High Explosives.....                                                                            | 8vo,         | 4 00   |
| Erdmann's Introduction to Chemical Preparations. (Dunlap.).....                                                  | 12mo,        | 1 25   |
| * Fischer's Physiology of Alimentation.....                                                                      | Large 12mo,  | 2 00   |
| Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe.                                    | 12mo, mor.   | 1 50   |
| Fowler's Sewage Works Analyses.....                                                                              | 12mo,        | 2 00   |
| Fresenius's Manual of Qualitative Chemical Analysis. (Wells.).....                                               | 8vo,         | 5 00   |
| Manual of Qualitative Chemical Analysis. Part I. Descriptive. (Wells.)                                           | 8vo,         | 3 00   |
| Quantitative Chemical Analysis. (Cohn.)                                                                          | 2 vols. 8vo, | 12 50  |
| When Sold Separately, Vol. I, \$6. Vol. II, \$8.                                                                 |              |        |
| Fuertes's Water and Public Health.....                                                                           | 12mo,        | 1 50   |
| Furman and Pardoe's Manual of Practical Assaying. (Sixth Edition, Revised and Enlarged.).....                    | 8vo,         | 3 00   |
| * Getman's Exercises in Physical Chemistry.....                                                                  | 12mo,        | 2 00   |
| Gill's Gas and Fuel Analysis for Engineers.....                                                                  | 12mo,        | 1 25   |
| * Gooch and Browning's Outlines of Qualitative Chemical Analysis.                                                | Large 12mo,  | 1 25   |
| Grotenfelt's Principles of Modern Dairy Practice. (Woll.).....                                                   | 12mo,        | 2 00   |
| Groth's Introduction to Chemical Crystallography (Marshall). ....                                                | 12mo,        | 1 25   |
| Hammarsten's Text-book of Physiological Chemistry. (Mandel.).....                                                | 8vo,         | 4 00   |
| Hanausek's Microscopy of Technical Products. (Winton.).....                                                      | 8vo,         | 5 00   |
| * Haskins and Macleod's Organic Chemistry.....                                                                   | 12mo,        | 2 00   |
| Hering's Ready Reference Tables (Conversion Factors).....                                                        | 16mo, mor.   | 2 50   |
| * Herrick's Denatured or Industrial Alcohol.....                                                                 | 8vo,         | 4 00   |
| Hinds's Inorganic Chemistry.....                                                                                 | 8vo,         | 3 00   |
| * Laboratory Manual for Students.....                                                                            | 12mo,        | 1 00   |
| * Holleman's Laboratory Manual of Organic Chemistry for Beginners. (Walker.).....                                | 12mo,        | 1 00   |
| Text-book of Inorganic Chemistry. (Cooper.).....                                                                 | 8vo,         | 2 50   |
| Text-book of Organic Chemistry. (Walker and Mott.).....                                                          | 8vo,         | 2 50   |
| * Holley's Lead and Zinc Pigments.....                                                                           | Large 12mo,  | 3 00   |
| Holley and Ladd's Analysis of Mixed Paints, Color Pigments, and Varnishes.                                       | Large 12mo,  | 2 50   |
| Hopkins's Oil-chemists' Handbook.....                                                                            | 8vo,         | 3 00   |
| Jackson's Directions for Laboratory Work in Physiological Chemistry..                                            | 8vo,         | 1 25   |
| Johnson's Rapid Methods for the Chemical Analysis of Special Steels, Steel-making Alloys and Graphite.....       | Large 12mo,  | 3 00   |
| Landauer's Spectrum Analysis. (Tingle.).....                                                                     | 8vo,         | 3 00   |
| * Langworthy and Austen's Occurrence of Aluminum in Vegetable Products, Animal Products, and Natural Waters..... | 8vo,         | 2 00   |
| Lassar-Cohn's Application of Some General Reactions to Investigations in Organic Chemistry. (Tingle.).....       | 12mo,        | 1 00   |
| Leach's Inspection and Analysis of Food with Special Reference to State Control.....                             | 8vo,         | 7 50   |
| Löb's Electrochemistry of Organic Compounds. (Lorenz.).....                                                      | 8vo,         | 3 00   |
| Lodge's Notes on Assaying and Metallurgical Laboratory Experiments.....                                          | 8vo,         | 3 00   |
| Low's Technical Method of Ore Analysis.....                                                                      | 8vo,         | 3 00   |

|                                                                             |            |        |
|-----------------------------------------------------------------------------|------------|--------|
| Lowe's Paint for Steel Structures. ....                                     | 12mo.      | \$1 00 |
| Lunge's Techno-chemical Analysis. (Cohn.).....                              | 12mo.      | 1 00   |
| * McKay and Larsen's Principles and Practice of Butter-making. ....         | 8vo.       | 1 50   |
| Maire's Modern Pigments and their Vehicles. ....                            | 12mo.      | 2 00   |
| Mandel's Handbook for Bio-chemical Laboratory. ....                         | 12mo.      | 1 50   |
| * Martin's Laboratory Guide to Qualitative Analysis with the Blowpipe       | 12mo.      | 0 60   |
| Mason's Examination of Water. (Chemical and Bacteriological.)....           | 12mo.      | 1 25   |
| Water-supply. (Considered Principally from a Sanitary Standpoint.)          | 8vo.       | 4 00   |
| * Mathewson's First Principles of Chemical Theory. ....                     | 8vo.       | 1 00   |
| Matthews's Laboratory Manual of Dyeing and Textile Chemistry. ....          | 8vo.       | 3 50   |
| Textile Fibres. 2d Edition, Rewritten. ....                                 | 8vo.       | 4 00   |
| * Meyer's Determination of Radicles in Carbon Compounds. (Tingle.)          | 12mo.      | 1 25   |
| Third Edition. ....                                                         | 12mo.      | 1 00   |
| Miller's Cyanide Process. ....                                              | 12mo.      | 1 00   |
| Manual of Assaying. ....                                                    | 12mo.      | 1 00   |
| Minet's Production of Aluminum and its Industrial Use. (Waldo.)....         | 12mo.      | 2 50   |
| * Mittelstaedt's Technical Calculations for Sugar Works. (Bourbakis.)       | 12mo.      | 1 50   |
| Mixer's Elementary Text-book of Chemistry. ....                             | 12mo.      | 1 50   |
| Morgan's Elements of Physical Chemistry. ....                               | 12mo.      | 3 00   |
| Outline of the Theory of Solutions and its Results. ....                    | 12mo.      | 1 00   |
| * Physical Chemistry for Electrical Engineers. ....                         | 12mo.      | 1 50   |
| Morse's Calculations used in Cane-sugar Factories. ....                     | 16mo. mor. | 1 50   |
| * Muir's History of Chemical Theories and Laws. ....                        | 8vo.       | 4 00   |
| Mulliken's General Method for the Identification of Pure Organic Compounds. |            |        |
| Vol. I. Compounds of Carbon with Hydrogen and Oxygen. Large 8vo.            |            | 5 00   |
| Vol. II. Nitrogenous Compounds. (In Preparation.)                           |            |        |
| Vol. III. The Commercial Dyestuffs. ....                                    | Large 8vo. | 5 00   |
| O'Driscoll's Notes on the Treatment of Gold Ores. ....                      | 8vo.       | 2 00   |
| Ostwald's Conversations on Chemistry. Part One. (Ramsey.)....               | 12mo.      | 1 50   |
| " " " " Part Two. (Turnbull.)....                                           | 12mo.      | 2 00   |
| Owen and Standage's Dyeing and Cleaning of Textile Fabrics. ....            | 12mo.      | 2 00   |
| * Palmer's Practical Test Book of Chemistry. ....                           | 12mo.      | 1 00   |
| * Pauli's Physical Chemistry in the Service of Medicine. (Fischer.)         | 12mo.      | 1 25   |
| Penfield's Tables of Minerals, Including the Use of Minerals and Statistics |            |        |
| of Domestic Production. ....                                                | 8vo.       | 1 00   |
| Pictet's Alkaloids and their Chemical Constitution. (Biddle.)....           | 8vo.       | 5 00   |
| Poole's Calorific Power of Fuels. ....                                      | 8vo.       | 3 00   |
| Prescott and Winslow's Elements of Water Bacteriology, with Special Refer-  |            |        |
| ence to Sanitary Water Analysis. ....                                       | 12mo.      | 1 50   |
| * Reisig's Guide to Piece-Dyeing. ....                                      | 8vo.       | 25 00  |
| Richards and Woodman's Air, Water, and Food from a Sanitary Stand-          |            |        |
| point. ....                                                                 | 8vo.       | 2 00   |
| Ricketts and Miller's Notes on Assaying. ....                               | 8vo.       | 3 00   |
| Rideal's Disinfection and the Preservation of Food. ....                    | 8vo.       | 4 00   |
| Sewage and the Bacterial Purification of Sewage. ....                       | 8vo.       | 4 00   |
| Rigg's Elementary Manual for the Chemical Laboratory. ....                  | 8vo.       | 1 25   |
| Robine and Lenglen's Cyanide Industry. (Le Clerc.)....                      | 8vo.       | 4 00   |
| Ruddiman's Incompatibilities in Prescriptions. ....                         | 8vo.       | 2 00   |
| Whys in Pharmacy. ....                                                      | 12mo.      | 1 00   |
| * Ruer's Elements of Metallography. (Mathewson.) ....                       | 8vo.       | 3 00   |
| Sabin's Industrial and Artistic Technology of Paint and Varnish. ....       | 8vo.       | 3 00   |
| Salkowski's Physiological and Pathological Chemistry. (Orndorff.)....       | 8vo.       | 2 50   |
| Schimpf's Essentials of Volumetric Analysis. ....                           | 12mo.      | 1 25   |
| Manual of Volumetric Analysis. (Fifth Edition, Rewritten). ....             | 8vo.       | 5 00   |
| * Qualitative Chemical Analysis. ....                                       | 8vo.       | 1 25   |
| Seamon's Manual for Assayers and Chemists. (In Press.)                      |            |        |
| Smith's Lecture Notes on Chemistry for Dental Students. ....                | 8vo.       | 2 50   |
| Spencer's Handbook for Cane Sugar Manufacturers. ....                       | 16mo. mor. | 3 00   |
| Handbook for Chemists of Beet-sugar Houses. ....                            | 16mo. mor. | 3 00   |
| Stockbridge's Rocks and Soils. ....                                         | 8vo.       | 2 50   |
| Stone's Practical Testing of Gas and Gas Meters. ....                       | 8vo.       | 3 50   |
| * Tillman's Descriptive General Chemistry. ....                             | 8vo.       | 3 00   |
| * Elementary Lessons in Heat                                                | 8vo.       | 1 50   |
| Treadwell's Qualitative Analysis. (Hall.)....                               | 8vo.       | 3 00   |
| Quantitative Analysis. (Hall.)....                                          | 8vo.       | 4 00   |



|                                                                                       |                    |        |
|---------------------------------------------------------------------------------------|--------------------|--------|
| Soper's Air and Ventilation of Subways.....                                           | 12mo,              | \$2 50 |
| * Tracy's Exercises in Surveying.....                                                 | 12mo, mor.         | 1 00   |
| Tracy's Plane Surveying.....                                                          | 16mo, mor.         | 3 00   |
| * Trautwine's Civil Engineer's Pocket-book.....                                       | 16mo, mor.         | 5 00   |
| Venable's Garbage Crematories in America.....                                         | 8vo,               | 2 00   |
| Methods and Devices for Bacterial Treatment of Sewage.....                            | 8vo,               | 3 00   |
| Wait's Engineering and Architectural Jurisprudence.....                               | 8vo,               | 6 00   |
| Sheep.....                                                                            |                    | 6 50   |
| Law of Contracts.....                                                                 | 8vo,               | 3 00   |
| Law of Operations Preliminary to Construction in Engineering and<br>Architecture..... | 8vo,               | 5 00   |
| Sheep.....                                                                            |                    | 5 50   |
| Warren's Stereotomy—Problems in Stone-cutting.....                                    | 8vo,               | 2 50   |
| * Waterbury's Vest-Pocket Hand-book of Mathematics for Engineers.....                 | 2½×5½ inches, mor. | 1 00   |
| * Enlarged Edition, Including Tables.....                                             | mor.               | 1 50   |
| Webb's Problem's in the Use and Adjustment of Engineering Instruments.....            | 16mo, mor.         | 1 25   |
| Wilson's Topographic Surveying.....                                                   | 8vo,               | 3 50   |

## BRIDGES AND ROOFS.

|                                                                                                         |               |       |
|---------------------------------------------------------------------------------------------------------|---------------|-------|
| Boller's Practical Treatise on the Construction of Iron Highway Bridges.....                            | 8vo,          | 2 00  |
| * Thames River Bridge.....                                                                              | Oblong paper, | 5 00  |
| Burr and Falk's Design and Construction of Metallic Bridges.....                                        | 8vo,          | 5 00  |
| Influence Lines for Bridge and Roof Computations.....                                                   | 8vo,          | 3 00  |
| Du Bois's Mechanics of Engineering. Vol. II.....                                                        | Small 4to,    | 10 00 |
| Foster's Treatise on Wooden Trestle Bridges.....                                                        | 4to,          | 5 00  |
| Fowler's Ordinary Foundations.....                                                                      | 8vo,          | 3 50  |
| Greene's Arches in Wood, Iron, and Stone.....                                                           | 8vo,          | 2 50  |
| Bridge Trusses.....                                                                                     | 8vo,          | 2 50  |
| Roof Trusses.....                                                                                       | 8vo,          | 1 25  |
| Grimm's Secondary Stresses in Bridge Trusses.....                                                       | 8vo,          | 2 50  |
| Heller's Stresses in Structures and the Accompanying Deformations....                                   | 8vo,          | 3 00  |
| Howe's Design of Simple Roof-trusses in Wood and Steel.....                                             | 8vo,          | 2 00  |
| Symmetrical Masonry Arches.....                                                                         | 8vo,          | 2 50  |
| Treatise on Arches.....                                                                                 | 8vo,          | 4 00  |
| * Jacoby's Structural Details, or Elements of Design in Heavy Framing, 8vo,                             |               | 2 25  |
| Johnson, Bryan and Turneaure's Theory and Practice in the Designing of<br>Modern Framed Structures..... | Small 4to,    | 10 00 |
| Merriman and Jacoby's Text-book on Roofs and Bridges:                                                   |               |       |
| Part I. Stresses in Simple Trusses.....                                                                 | 8vo,          | 2 50  |
| Part II. Graphic Statics.....                                                                           | 8vo,          | 2 50  |
| Part III. Bridge Design.....                                                                            | 8vo,          | 2 50  |
| Part IV. Higher Structures.....                                                                         | 8vo,          | 2 50  |
| Morison's Memphis Bridge.....                                                                           | Oblong 4to,   | 10 00 |
| Sondericker's Graphic Statics, with Applications to Trusses, Beams, and<br>Arches.....                  | 8vo,          | 2 00  |
| Waddell's De Pontibus, Pocket-book for Bridge Engineers.....                                            | 16mo, mor.    | 2 00  |
| * Specifications for Steel Bridges.....                                                                 | 12mo,         | 50    |
| Waddell and Harrington's Bridge Engineering. (In Preparation.)                                          |               |       |
| Wright's Designing of Draw-spans. Two parts in one volume.....                                          | 8vo,          | 3 50  |

## HYDRAULICS.

|                                                                                                           |                    |      |
|-----------------------------------------------------------------------------------------------------------|--------------------|------|
| Barnes's Ice Formation.....                                                                               | 8vo,               | 3 00 |
| Bazin's Experiments upon the Contraction of the Liquid Vein Issuing from<br>an Orifice. (Trautwine.)..... | 8vo,               | 2 00 |
| Bovey's Treatise on Hydraulics.....                                                                       | 8vo,               | 5 00 |
| Church's Diagrams of Mean Velocity of Water in Open Channels.....                                         | Oblong 4to, paper, | 1 50 |
| Hydraulic Motors.....                                                                                     | 8vo,               | 2 00 |

|                                                                                                                                  |             |        |
|----------------------------------------------------------------------------------------------------------------------------------|-------------|--------|
| Coffin's Graphical Solution of Hydraulic Problems.....                                                                           | 16mo, mor.  | \$2 50 |
| Flather's Dynamometers, and the Measurement of Power.....                                                                        | 12mo,       | 3 00   |
| Folwell's Water-supply Engineering.....                                                                                          | 8vo,        | 4 00   |
| Frizell's Water-power.....                                                                                                       | 8vo,        | 5 00   |
| Fuertes's Water and Public Health.....                                                                                           | 12mo,       | 1 50   |
| Water-filtration Works.....                                                                                                      | 12mo,       | 2 50   |
| Ganguillet and Kutter's General Formula for the Uniform Flow of Water in Rivers and Other Channels. (Hering and Trautwine.)..... | 8vo,        | 4 00   |
| Hazen's Clean Water and How to Get It.....                                                                                       | Large 12mo, | 1 50   |
| Filtration of Public Water-supplies.....                                                                                         | 8vo,        | 3 00   |
| Hazelhurst's Towers and Tanks for Water-works.....                                                                               | 8vo,        | 2 50   |
| Herschel's 115 Experiments on the Carrying Capacity of Large, Riveted, Metal Conduits.....                                       | 8vo,        | 2 00   |
| Hoyt and Grover's River Discharge.....                                                                                           | 8vo,        | 2 00   |
| Hubbard and Kiersted's Water-works Management and Maintenance.....                                                               | 8vo,        | 4 00   |
| * Lyndon's Development and Electrical Distribution of Water Power.....                                                           | 8vo,        | 3 00   |
| Mason's Water-supply. (Considered Principally from a Sanitary Stand-point.).....                                                 | 8vo,        | 4 00   |
| Merriman's Treatise on Hydraulics.....                                                                                           | 8vo,        | 5 00   |
| * Molitor's Hydraulics of Rivers, Weirs and Sluices.....                                                                         | 8vo,        | 2 00   |
| * Richards's Laboratory Notes on Industrial Water Analysis.....                                                                  | 8vo,        | 50     |
| Schuyler's Reservoirs for Irrigation, Water-power, and Domestic Water-supply. Second Edition, Revised and Enlarged.....          | Large 8vo,  | 6 00   |
| * Thomas and Watt's Improvement of Rivers.....                                                                                   | 4to,        | 6 00   |
| Turneure and Russell's Public Water-supplies.....                                                                                | 8vo,        | 5 00   |
| Wegmann's Design and Construction of Dams. 5th Ed., enlarged.....                                                                | 4to,        | 6 00   |
| Water-Supply of the City of New York from 1658 to 1895.....                                                                      | 4to,        | 10 00  |
| Whipple's Value of Pure Water.....                                                                                               | Large 12mo, | 1 00   |
| Williams and Hazen's Hydraulic Tables.....                                                                                       | 8vo,        | 1 50   |
| Wilson's Irrigation Engineering.....                                                                                             | Large 8vo,  | 4 00   |
| Wood's Turbines.....                                                                                                             | 8vo,        | 2 50   |

## MATERIALS OF ENGINEERING.

|                                                                                                                    |             |       |
|--------------------------------------------------------------------------------------------------------------------|-------------|-------|
| Baker's Roads and Pavements.....                                                                                   | 8vo,        | 5 00  |
| Treatise on Masonry Construction.....                                                                              | 8vo,        | 5 00  |
| Black's United States Public Works.....                                                                            | Oblong 4to, | 5 00  |
| Blanchard's Bituminous Roads. (In Press.)                                                                          |             |       |
| Bleining's Manufacture of Hydraulic Cement. (In Preparation.)                                                      |             |       |
| * Bovey's Strength of Materials and Theory of Structures.....                                                      | 8vo,        | 7 50  |
| Burr's Elasticity and Resistance of the Materials of Engineering.....                                              | 8vo,        | 7 50  |
| Byrne's Highway Construction.....                                                                                  | 8vo,        | 5 00  |
| Inspection of the Materials and Workmanship Employed in Construction.....                                          | 16mo,       | 3 00  |
| Church's Mechanics of Engineering.....                                                                             | 8vo,        | 6 00  |
| Du Bois's Mechanics of Engineering.                                                                                |             |       |
| Vol. I. Kinematics, Statics, Kinetics.....                                                                         | Small 4to,  | 7 50  |
| Vol. II. The Stresses in Framed Structures, Strength of Materials and Theory of Flexures.....                      | Small 4to,  | 10 00 |
| * Eckel's Cements, Limes, and Plasters.....                                                                        | 8vo,        | 6 00  |
| Stone and Clay Products used in Engineering. (In Preparation.)                                                     |             |       |
| Fowler's Ordinary Foundations.....                                                                                 | 8vo,        | 3 50  |
| * Greene's Structural Mechanics.....                                                                               | 8vo,        | 2 50  |
| * Holley's Lead and Zinc Pigments.....                                                                             | Large 12mo, | 3 00  |
| Holley and Ladd's Analysis of Mixed Paints, Color Pigments and Varnishes.....                                      | Large 12mo, | 2 50  |
| Hubbard's Dust Preventives and Road Binders. (In Preparation.)                                                     |             |       |
| Johnson's (C. M.) Rapid Methods for the Chemical Analysis of Special Steels, Steel-making Alloys and Graphite..... | Large 12mo, | 3 00  |
| Johnson's (J. B.) Materials of Construction.....                                                                   | Large 8vo,  | 6 00  |
| Keep's Cast Iron.....                                                                                              | 8vo,        | 2 50  |
| Lanza's Applied Mechanics.....                                                                                     | 8vo,        | 7 50  |
| Lowe's Paint for Steel Structures.....                                                                             | 12mo,       | 1 00  |

|                                                                                                                 |            |        |
|-----------------------------------------------------------------------------------------------------------------|------------|--------|
| Maire's Modern Pigments and their Vehicles.....                                                                 | 12mo,      | \$2 00 |
| Martens's Handbook on Testing Materials. (Henning.) 2 vols.....                                                 | 8vo,       | 7 50   |
| Maurer's Technical Mechanics.....                                                                               | 8vo,       | 4 00   |
| Merrill's Stones for Building and Decoration.....                                                               | 8vo,       | 5 00   |
| Merriman's Mechanics of Materials.....                                                                          | 8vo,       | 5 00   |
| * Strength of Materials.....                                                                                    | 12mo,      | 1 00   |
| Metcalf's Steel. A Manual for Steel-users.....                                                                  | 12mo,      | 2 00   |
| Morrison's Highway Engineering.....                                                                             | 8vo,       | 2 50   |
| Patton's Practical Treatise on Foundations.....                                                                 | 8vo,       | 5 00   |
| Rice's Concrete Block Manufacture.....                                                                          | 8vo,       | 2 00   |
| Richardson's Modern Asphalt Pavements.....                                                                      | 8vo,       | 3 00   |
| Richey's Building Foreman's Pocket Book and Ready Reference.....                                                | 16mo, mor. | 5 00   |
| * Cement Workers' and Plasterers' Edition (Building Mechanics' Ready Reference Series).....                     | 16mo, mor. | 1 50   |
| Handbook for Superintendents of Construction.....                                                               | 16mo, mor. | 4 00   |
| * Stone and Brick Masons' Edition (Building Mechanics' Ready Reference Series).....                             | 16mo, mor. | 1 50   |
| * Ries's Clays: Their Occurrence, Properties, and Uses.....                                                     | 8vo,       | 5 00   |
| * Ries and Leighton's History of the Clay-working Industry of the United States.....                            | 8vo,       | 2 50   |
| Sabin's Industrial and Artistic Technology of Paint and Varnish.....                                            | 8vo,       | 3 00   |
| * Smith's Strength of Material.....                                                                             | 12mo,      | 1 25   |
| Snow's Principal Species of Wood.....                                                                           | 8vo,       | 3 50   |
| Spalding's Hydraulic Cement.....                                                                                | 12mo,      | 2 00   |
| Text-book on Roads and Pavements.....                                                                           | 12mo,      | 2 00   |
| Taylor and Thompson's Treatise on Concrete, Plain and Reinforced....                                            | 8vo,       | 5 00   |
| Thurston's Materials of Engineering. In Three Parts.....                                                        | 8vo,       | 8 00   |
| Part I. Non-metallic Materials of Engineering and Metallurgy....                                                | 8vo,       | 2 00   |
| Part II. Iron and Steel.....                                                                                    | 8vo,       | 3 50   |
| Part III. A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents.....                          | 8vo,       | 2 50   |
| Tillson's Street Pavements and Paving Materials.....                                                            | 8vo,       | 4 00   |
| * Trautwine's Concrete, Plain and Reinforced.....                                                               | 16mo,      | 2 00   |
| Turneure and Maurer's Principles of Reinforced Concrete Construction. Second Edition, Revised and Enlarged..... | 8vo,       | 3 50   |
| Waterbury's Cement Laboratory Manual.....                                                                       | 12mo,      | 1 00   |
| Wood's (De V.) Treatise on the Resistance of Materials, and an Appendix on the Preservation of Timber.....      | 8vo,       | 2 00   |
| Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel.....                             | 8vo,       | 4 00   |

#### RAILWAY ENGINEERING.

|                                                                                        |                  |      |
|----------------------------------------------------------------------------------------|------------------|------|
| Andrews's Handbook for Street Railway Engineers.....                                   | 3X5 inches, mor. | 1 25 |
| Berg's Buildings and Structures of American Railroads.....                             | 4to,             | 5 00 |
| Brooks's Handbook of Street Railroad Location.....                                     | 16mo, mor.       | 1 50 |
| Butts's Civil Engineer's Field-book.....                                               | 16mo, mor.       | 2 50 |
| Crandall's Railway and Other Earthwork Tables.....                                     | 8vo,             | 1 50 |
| Transition Curve.....                                                                  | 16mo, mor.       | 1 50 |
| * Crockett's Methods for Earthwork Computations.....                                   | 8vo,             | 1 50 |
| Dredge's History of the Pennsylvania Railroad. (1879).....                             | Paper            | 5 00 |
| Fisher's Table of Cubic Yards.....                                                     | Cardboard,       | 25   |
| Godwin's Railroad Engineers' Field-book and Explorers' Guide.....                      | 16mo, mor.       | 2 50 |
| Hudson's Tables for Calculating the Cubic Contents of Excavations and Embankments..... | 8vo,             | 1 00 |
| Ives and Hilts's Problems in Surveying, Railroad Surveying and Geodesy.....            | 16mo, mor.       | 1 50 |
| Molitor and Beard's Manual for Resident Engineers.....                                 | 16mo,            | 1 00 |
| Nagle's Field Manual for Railroad Engineers.....                                       | 16mo, mor.       | 3 00 |
| * Orrock's Railroad Structures and Estimates.....                                      | 8vo,             | 3 00 |
| Philbrick's Field Manual for Engineers.....                                            | 16mo, mor.       | 3 00 |
| Raymond's Railroad Engineering. 3 volumes.                                             |                  |      |
| Vol. I. Railroad Field Geometry. (In Preparation.)                                     |                  |      |
| Vol. II. Elements of Railroad Engineering.....                                         | 8vo,             | 3 50 |
| Vol. III. Railroad Engineer's Field Book. (In Preparation.)                            |                  |      |

|                                                                                                       |             |        |
|-------------------------------------------------------------------------------------------------------|-------------|--------|
| Searles's Field Engineering.....                                                                      | 16mo, mor.  | \$3 00 |
| Railroad Spiral.....                                                                                  | 16mo, mor.  | 1 50   |
| Taylor's Prismatical Formulae and Earthwork.....                                                      | 8vo,        | 1 50   |
| * Trautwine's Field Practice of Laying Out Circular Curves for Railroads.....                         | 12mo, mor.  | 2 50   |
| * Method of Calculating the Cubic Contents of Excavations and Embankments by the Aid of Diagrams..... | 8vo,        | 2 00   |
| Webb's Economics of Railroad Construction.....                                                        | Large 12mo, | 2 50   |
| Railroad Construction.....                                                                            | 16mo, mor.  | 5 00   |
| Wellington's Economic Theory of the Location of Railways.....                                         | Large 12mo, | 5 00   |
| Wilson's Elements of Railroad-Track and Construction.....                                             | 12mo,       | 2 00   |

## DRAWING.

|                                                                                   |             |      |
|-----------------------------------------------------------------------------------|-------------|------|
| Barr's Kinematics of Machinery.....                                               | 8vo,        | 2 50 |
| * Bartlett's Mechanical Drawing.....                                              | 8vo,        | 3 00 |
| * " " " " Abridged Ed.....                                                        | 8vo,        | 1 50 |
| Coolidge's Manual of Drawing.....                                                 | 8vo, paper, | 1 00 |
| Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers..... | Oblong 4to, | 2 50 |
| Durley's Kinematics of Machines.....                                              | 8vo,        | 4 00 |
| Emch's Introduction to Projective Geometry and its Application.....               | 8vo,        | 2 50 |
| French and Ives' Stereotomy.....                                                  | 8vo,        | 2 50 |
| Hill's Text-book on Shades and Shadows, and Perspective.....                      | 8vo,        | 2 00 |
| Jamison's Advanced Mechanical Drawing.....                                        | 8vo,        | 2 00 |
| Elements of Mechanical Drawing.....                                               | 8vo,        | 2 50 |
| Jones's Machine Design:                                                           |             |      |
| Part I. Kinematics of Machinery.....                                              | 8vo,        | 1 50 |
| Part II. Form, Strength, and Proportions of Parts.....                            | 8vo,        | 3 00 |
| Kimball and Barr's Machine Design.....                                            | 8vo,        | 3 00 |
| MacCord's Elements of Descriptive Geometry.....                                   | 8vo,        | 3 00 |
| Kinematics; or, Practical Mechanism.....                                          | 8vo,        | 5 00 |
| Mechanical Drawing.....                                                           | 4to,        | 4 00 |
| Velocity Diagrams.....                                                            | 8vo,        | 1 50 |
| McLeod's Descriptive Geometry.....                                                | Large 12mo, | 1 50 |
| * Mahan's Descriptive Geometry and Stone-cutting.....                             | 8vo,        | 1 50 |
| Industrial Drawing. (Thompson.).....                                              | 8vo,        | 3 50 |
| Moyer's Descriptive Geometry.....                                                 | 8vo,        | 2 00 |
| Reed's Topographical Drawing and Sketching.....                                   | 4to,        | 5 00 |
| Reid's Course in Mechanical Drawing.....                                          | 8vo,        | 2 00 |
| Text book of Mechanical Drawing and Elementary Machine Design.....                | 8vo,        | 3 00 |
| Robinson's Principles of Mechanism.....                                           | 8vo,        | 3 00 |
| Schwamb and Merrill's Elements of Mechanism.....                                  | 8vo,        | 3 00 |
| Smith (A. W.) and Marx's Machine Design.....                                      | 8vo,        | 3 00 |
| Smith's (R. S.) Manual of Topographical Drawing. (McMillan.).....                 | 8vo,        | 2 50 |
| * Titsworth's Elements of Mechanical Drawing.....                                 | Oblong 8vo, | 1 25 |
| Warren's Drafting Instruments and Operations.....                                 | 12mo,       | 1 25 |
| Elements of Descriptive Geometry, Shadows, and Perspective.....                   | 8vo,        | 3 50 |
| Elements of Machine Construction and Drawing.....                                 | 8vo,        | 7 50 |
| Elements of Plane and Solid Free-hand Geometrical Drawing.....                    | 12mo,       | 1 00 |
| General Problems of Shades and Shadows.....                                       | 8vo,        | 3 00 |
| Manual of Elementary Problems in the Linear Perspective of Forms and Shadow.....  | 12mo,       | 1 00 |
| Manual of Elementary Projection Drawing.....                                      | 12mo,       | 1 50 |
| Plane Problems in Elementary Geometry.....                                        | 12mo,       | 1 25 |
| Problems, Theorems, and Examples in Descriptive Geometry.....                     | 8vo,        | 2 50 |
| Weisbach's Kinematics and Power of Transmission. (Hermann and Klein.).....        | 8vo,        | 5 00 |
| Wilson's (H. M.) Topographic Surveying.....                                       | 8vo,        | 3 50 |
| * Wilson's (V. T.) Descriptive Geometry.....                                      | 8vo,        | 1 50 |
| Free-hand Lettering.....                                                          | 8vo,        | 1 00 |
| Free-hand Perspective.....                                                        | 8vo,        | 2 50 |
| Woelf's Elementary Course in Descriptive Geometry.....                            | Large 8vo,  | 3 00 |

## ELECTRICITY AND PHYSICS.

|                                                                                                                     |                  |        |
|---------------------------------------------------------------------------------------------------------------------|------------------|--------|
| * Abegg's Theory of Electrolytic Dissociation. (von Ende.).....                                                     | 12mo,            | \$1 25 |
| Andrews's Hand-book for Street Railway Engineering.....                                                             | 3×5 inches, mor. | 1 25   |
| Anthony and Brackett's Text-book of Physics. (Magie.)....                                                           | Large 12mo,      | 3 00   |
| Anthony and Ball's Lecture-notes on the Theory of Electrical Measure-<br>ments.....                                 | 12mo,            | 1 00   |
| Benjamin's History of Electricity.....                                                                              | .8vo,            | 3 00   |
| Voltaic Cell.....                                                                                                   | .8vo,            | 3 00   |
| Betts's Lead Refining and Electrolysis.....                                                                         | .8vo,            | 4 00   |
| Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.)                                               | .8vo,            | 3 00   |
| * Collins's Manual of Wireless Telegraphy and Telephony.....                                                        | 12mo,            | 1 50   |
| Crehore and Squier's Polarizing Photo-chronograph.....                                                              | .8vo,            | 3 00   |
| * Danneel's Electrochemistry. (Merriam.).....                                                                       | 12mo,            | 1 25   |
| Dawson's "Engineering" and Electric Traction Pocket-book....                                                        | 16mo, mor.       | 5 00   |
| Dolezalek's Theory of the Lead Accumulator (Storage Battery). (von Ende.)                                           | 12mo,            | 2 50   |
| Duhem's Thermodynamics and Chemistry. (Burgess.).....                                                               | .8vo,            | 4 00   |
| Flather's Dynamometers, and the Measurement of Power.....                                                           | 12mo,            | 3 00   |
| Getman's Introduction to Physical Science.....                                                                      | 12mo,            | 1 50   |
| Gilbert's De Magnete. (Mottelay).....                                                                               | .8vo,            | 2 50   |
| * Hanchett's Alternating Currents.....                                                                              | 12mo,            | 1 00   |
| Hering's Ready Reference Tables (Conversion Factors).....                                                           | 16mo, mor.       | 2 50   |
| * Hobart and Ellis's High-speed Dynamo Electric Machinery.....                                                      | .8vo,            | 6 00   |
| Holman's Precision of Measurements.....                                                                             | .8vo,            | 2 00   |
| Telescopic Mirror-scale Method, Adjustments, and Tests....                                                          | Large 8vo,       | 75     |
| * Karapetoff's Experimental Electrical Engineering.....                                                             | .8vo,            | 6 00   |
| Kinzbrunner's Testing of Continuous-current Machines.....                                                           | .8vo,            | 2 00   |
| Landauer's Spectrum Analysis. (Tingle.).....                                                                        | .8vo,            | 3 00   |
| Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.)                                                  | 12mo,            | 3 00   |
| Löb's Electrochemistry of Organic Compounds. (Lorenz.).....                                                         | .8vo,            | 3 00   |
| * Lyndon's Development and Electrical Distribution of Water Power.                                                  | .8vo,            | 3 00   |
| * Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II.                                                   | .8vo, each,      | 6 00   |
| * Michie's Elements of Wave Motion Relating to Sound and Light....                                                  | .8vo,            | 4 00   |
| Morgan's Outline of the Theory of Solution and its Results.....                                                     | 12mo,            | 1 00   |
| * Physical Chemistry for Electrical Engineers.....                                                                  | 12mo,            | 1 50   |
| * Norris's Introduction to the Study of Electrical Engineering.....                                                 | .8vo,            | 2 50   |
| Norris and Dennison's Course of Problems on the Electrical Characteristics of<br>Circuits and Machines. (In Press.) |                  |        |
| * Parshall and Hobart's Electric Machine Design.....                                                                | 4to, half mor.   | 12 50  |
| Reagan's Locomotives: Simple, Compound, and Electric. New Edition.                                                  | Large 12mo,      | 3 50   |
| * Rosenberg's Electrical Engineering. (Haldane Gee—Kinzbrunner.)                                                    | .8vo,            | 2 00   |
| Ryan, Norris, and Hoxie's Electrical Machinery. Vol. I.....                                                         | .8vo,            | 2 50   |
| Schapper's Laboratory Guide for Students in Physical Chemistry....                                                  | 12mo,            | 1 00   |
| * Tillman's Elementary Lessons in Heat.....                                                                         | .8vo,            | 1 50   |
| Tory and Pitcher's Manual of Laboratory Physics.....                                                                | Large 12mo,      | 2 00   |
| Ulke's Modern Electrolytic Copper Refining.....                                                                     | .8vo,            | 3 00   |

## LAW.

|                                                                                       |             |      |
|---------------------------------------------------------------------------------------|-------------|------|
| * Brennan's Hand-book of Useful Legal Information for Business Men.                   | 16mo, mor.  | 5 00 |
| * Davis's Elements of Law.....                                                        | .8vo,       | 2 50 |
| * Treatise on the Military Law of United States.....                                  | .8vo,       | 7 00 |
| * Dudley's Military Law and the Procedure of Courts-martial.                          | Large 12mo, | 2 50 |
| Manual for Courts-martial.....                                                        | 16mo, mor.  | 1 50 |
| Wait's Engineering and Architectural Jurisprudence.....                               | .8vo,       | 6 00 |
|                                                                                       | Sheep,      | 6 50 |
| Law of Contracts.....                                                                 | .8vo,       | 3 00 |
| Law of Operations Preliminary to Construction in Engineering and<br>Architecture..... | .8vo,       | 5 00 |
|                                                                                       | Sheep,      | 5 50 |

# MATHEMATICS.

|                                                                              |              |        |
|------------------------------------------------------------------------------|--------------|--------|
| Baker's Elliptic Functions.....                                              | 8vo,         | \$1 50 |
| Briggs's Elements of Plane Analytic Geometry. (Böcher.) .....                | 12mo,        | 1 00   |
| * Buchanan's Plane and Spherical Trigonometry.....                           | 8vo,         | 1 00   |
| Byerley's Harmonic Functions.....                                            | 8vo,         | 1 00   |
| Chandler's Elements of the Infinitesimal Calculus.....                       | 12mo,        | 2 00   |
| * Coffin's Vector Analysis.....                                              | 12mo,        | 2 50   |
| Compton's Manual of Logarithmic Computations.....                            | 12mo,        | 1 50   |
| * Dickson's College Algebra.....                                             | Large 12mo,  | 1 50   |
| * Introduction to the Theory of Algebraic Equations.....                     | Large 12mo,  | 1 25   |
| Emch's Introduction to Projective Geometry and its Application.....          | 8vo,         | 2 50   |
| Fiske's Functions of a Complex Variable.....                                 | 8vo,         | 1 00   |
| Halsted's Elementary Synthetic Geometry.....                                 | 8vo,         | 1 50   |
| Elements of Geometry.....                                                    | 8vo,         | 1 75   |
| * Rational Geometry.....                                                     | 12mo,        | 1 50   |
| Synthetic Projective Geometry.....                                           | 8vo,         | 1 00   |
| Hyde's Grassmann's Space Analysis.....                                       | 8vo,         | 1 00   |
| * Johnson's (J. B.) Three-place Logarithmic Tables: Vest-pocket size, paper, |              | 15     |
| * 100 copies,                                                                |              | 5 00   |
| * Mounted on heavy cardboard, 8×10 inches,                                   |              | 25     |
| * 10 copies,                                                                 |              | 2 00   |
| Johnson's (W. W.) Abridged Editions of Differential and Integral Calculus.   |              |        |
| Large 12mo, 1 vol.                                                           |              | 2 50   |
| Curve Tracing in Cartesian Co-ordinates.....                                 | 12mo,        | 1 00   |
| Differential Equations.....                                                  | 8vo,         | 1 00   |
| Elementary Treatise on Differential Calculus.....                            | Large 12mo,  | 1 50   |
| Elementary Treatise on the Integral Calculus.....                            | Large 12mo,  | 1 50   |
| * Theoretical Mechanics.....                                                 | 12mo,        | 3 00   |
| Theory of Errors and the Method of Least Squares.....                        | 12mo,        | 1 50   |
| Treatise on Differential Calculus.....                                       | Large 12mo,  | 3 00   |
| Treatise on the Integral Calculus.....                                       | Large 12mo,  | 3 00   |
| Treatise on Ordinary and Partial Differential Equations... Large 12mo,       |              | 3 50   |
| Karapetoff's Engineering Applications of Higher Mathematics.                 |              |        |
| (In Preparation.)                                                            |              |        |
| Laplace's Philosophical Essay on Probabilities. (Truscott and Emory.) .....  | 12mo,        | 2 00   |
| * Ludlow and Bass's Elements of Trigonometry and Logarithmic and Other       |              |        |
| Tables.....                                                                  | 8vo,         | 3 00   |
| * Trigonometry and Tables published separately.....                          | Each,        | 2 00   |
| * Ludlow's Logarithmic and Trigonometric Tables.....                         | 8vo,         | 1 00   |
| Macfarlane's Vector Analysis and Quaternions.....                            | 8vo,         | 1 00   |
| McMahon's Hyperbolic Functions.....                                          | 8vo,         | 1 00   |
| Manning's Irrational Numbers and their Representation by Sequences and       |              |        |
| Series.....                                                                  | 12mo,        | 1 25   |
| Mathematical Monographs. Edited by Mansfield Merriman and Robert             |              |        |
| S. Woodward.....                                                             | Octavo, each | 1 00   |
| No. 1. History of Modern Mathematics, by David Eugene Smith.                 |              |        |
| No. 2. Synthetic Projective Geometry, by George Bruce Halsted.               |              |        |
| No. 3. Determinants, by Laenas Gifford Weld. No. 4. Hyper-                   |              |        |
| bolic Functions, by James McMahon. No. 5. Harmonic Func-                     |              |        |
| tions, by William E. Byerly. No. 6. Grassmann's Space Analysis,              |              |        |
| by Edward W. Hyde. No. 7. Probability and Theory of Errors,                  |              |        |
| by Robert S. Woodward. No. 8. Vector Analysis and Quaternions,               |              |        |
| by Alexander Macfarlane. No. 9. Differential Equations, by                   |              |        |
| William Woolsey Johnson. No. 10. The Solution of Equations,                  |              |        |
| by Mansfield Merriman. No. 11. Functions of a Complex Variable,              |              |        |
| by Thomas S. Fiske.                                                          |              |        |
| Maurer's Technical Mechanics.....                                            | 8vo,         | 4 00   |
| Merriman's Method of Least Squares.....                                      | 8vo,         | 2 00   |
| Solution of Equations.....                                                   | 8vo,         | 1 00   |
| Rice and Johnson's Differential and Integral Calculus. 2 vols. in one.       |              |        |
| Large 12mo,                                                                  |              | 1 50   |
| Elementary Treatise on the Differential Calculus.....                        | Large 12mo,  | 3 00   |
| Smith's History of Modern Mathematics.....                                   | 8vo,         | 1 00   |
| * Veblen and Lennes's Introduction to the Real Infinitesimal Analysis of One |              |        |
| Variable.....                                                                | 8vo,         | 2 00   |

|                                                                   |                                                |        |
|-------------------------------------------------------------------|------------------------------------------------|--------|
| * Waterbury's Vest Pocket Hand-book of Mathematics for Engineers. | 2 $\frac{1}{8}$ × 5 $\frac{1}{8}$ inches, mor. | \$1 00 |
| * Enlarged Edition, Including Tables .....                        | mor.                                           | 1 50   |
| Weld's Determinants. ....                                         | 8vo,                                           | 1 00   |
| Wood's Elements of Co-ordinate Geometry. ....                     | 8vo,                                           | 2 00   |
| Woodward's Probability and Theory of Errors. ....                 | 8vo,                                           | 1 00   |

## MECHANICAL ENGINEERING.

### MATERIALS OF ENGINEERING, STEAM-ENGINES AND BOILERS.

|                                                                                    |             |       |
|------------------------------------------------------------------------------------|-------------|-------|
| Bacon's Forge Practice. ....                                                       | 12mo,       | 1 50  |
| Baldwin's Steam Heating for Buildings. ....                                        | 12mo,       | 2 50  |
| Barr's Kinematics of Machinery. ....                                               | 8vo,        | 2 50  |
| * Bartlett's Mechanical Drawing. ....                                              | 8vo,        | 3 00  |
| * " " " Abridged Ed. ....                                                          | 8vo,        | 1 50  |
| * Burr's Ancient and Modern Engineering and the Isthmian Canal. ....               | 8vo,        | 3 50  |
| Carpenter's Experimental Engineering. ....                                         | 8vo,        | 6 00  |
| Heating and Ventilating Buildings. ....                                            | 8vo,        | 4 00  |
| Clerk's Gas and Oil Engine. (New edition in press.)                                |             |       |
| Compton's First Lessons in Metal Working. ....                                     | 12mo,       | 1 50  |
| Compton and De Groodt's Speed Lathe. ....                                          | 12mo,       | 1 50  |
| Coolidge's Manual of Drawing. ....                                                 | 8vo, paper, | 1 00  |
| Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers. .... | Oblong 4to, | 2 50  |
| Cromwell's Treatise on Belts and Pulleys. ....                                     | 12mo,       | 1 50  |
| Treatise on Toothed Gearing. ....                                                  | 12mo,       | 1 50  |
| Dingey's Machinery Pattern Making. ....                                            | 12mo,       | 2 00  |
| Durley's Kinematics of Machines. ....                                              | 8vo,        | 4 00  |
| Flanders's Gear-cutting Machinery. ....                                            | Large 12mo, | 3 00  |
| Fletcher's Dynamometers and the Measurement of Power. ....                         | 12mo,       | 3 00  |
| Rope Driving. ....                                                                 | 12mo,       | 2 00  |
| Gill's Gas and Fuel Analysis for Engineers. ....                                   | 12mo,       | 1 25  |
| Goss's Locomotive Sparks. ....                                                     | 8vo,        | 2 00  |
| Greene's Pumping Machinery. (In Preparation.)                                      |             |       |
| Hering's Ready Reference Tables (Conversion Factors). ....                         | 16mo, mor.  | 2 50  |
| * Hobart and Ellis's High Speed Dynamo Electric Machinery. ....                    | 8vo,        | 6 00  |
| Hutton's Gas Engine. ....                                                          | 8vo,        | 5 00  |
| Jamison's Advanced Mechanical Drawing. ....                                        | 8vo,        | 2 00  |
| Elements of Mechanical Drawing. ....                                               | 8vo,        | 2 50  |
| Jones's Gas Engine. ....                                                           | 8vo,        | 4 00  |
| Machine Design:                                                                    |             |       |
| Part I. Kinematics of Machinery. ....                                              | 8vo,        | 1 50  |
| Part II. Form, Strength, and Proportions of Parts. ....                            | 8vo,        | 3 00  |
| Kent's Mechanical Engineer's Pocket-Book. ....                                     | 16mo, mor.  | 5 00  |
| Kerr's Power and Power Transmission. ....                                          | 8vo,        | 2 00  |
| Kimball and Barr's Machine Design. ....                                            | 8vo,        | 3 00  |
| * Levin's Gas Engine. ....                                                         | 8vo,        | 4 00  |
| Leonard's Machine Shop Tools and Methods. ....                                     | 8vo,        | 4 00  |
| * Lorenz's Modern Refrigerating Machinery. (Pope, Haven, and Dean). ....           | 8vo,        | 4 00  |
| MacCord's Kinematics; or, Practical Mechanism. ....                                | 8vo,        | 5 00  |
| Mechanical Drawing. ....                                                           | 4to,        | 4 00  |
| Velocity Diagrams. ....                                                            | 8vo,        | 1 50  |
| MacFarland's Standard Reduction Factors for Gases. ....                            | 8vo,        | 1 50  |
| Mahan's Industrial Drawing. (Thompson). ....                                       | 8vo,        | 3 50  |
| Mehrtens's Gas Engine Theory and Design. ....                                      | Large 12mo, | 2 50  |
| Oberg's Handbook of Small Tools. ....                                              | Large 12mo, | 3 00  |
| * Parshall and Hobart's Electric Machine Design. Small 4to, half leather,          |             | 12 50 |
| Peele's Compressed Air Plant for Mines. ....                                       | 8vo,        | 3 00  |
| Poole's Calorific Power of Fuels. ....                                             | 8vo,        | 3 00  |
| * Porter's Engineering Reminiscences, 1855 to 1882. ....                           | 8vo,        | 3 00  |
| Reid's Course in Mechanical Drawing. ....                                          | 8vo,        | 2 00  |
| Text-book of Mechanical Drawing and Elementary Machine Design. ....                | 8vo,        | 3 00  |

|                                                                                     |                                                |        |
|-------------------------------------------------------------------------------------|------------------------------------------------|--------|
| Richards' Compressed Air. ....                                                      | 12mo,                                          | \$1 50 |
| Robinson's Principles of Mechanism. ....                                            | 8vo,                                           | 3 00   |
| Schwamb and Merrill's Elements of Mechanism. ....                                   | 8vo,                                           | 3 00   |
| Smith (A. W.) and Marx's Machine Design. ....                                       | 8vo,                                           | 3 00   |
| Smith's (O.) Press-working of Metals. ....                                          | 8vo,                                           | 3 00   |
| Sorel's Carbureting and Combustion in Alcohol Engines. (Woodward and Preston.) .... | Large 12mo,                                    | 3 00   |
| Stone's Practical Testing of Gas and Gas Meters. ....                               | 8vo,                                           | 3 50   |
| Thurston's Animal as a Machine and Prime Motor, and the Laws of Energetics. ....    | 12mo,                                          | 1 00   |
| Treatise on Friction and Lost Work in Machinery and Mill Work. ....                 | 8vo,                                           | 3 00   |
| * Tillson's Complete Automobile Instructor. ....                                    | 16mo,                                          | 1 50   |
| * Tittsworth's Elements of Mechanical Drawing. ....                                 | Oblong 8vo,                                    | 1 25   |
| Warren's Elements of Machine Construction and Drawing. ....                         | 8vo,                                           | 7 50   |
| * Waterbury's Vest Pocket Hand-book of Mathematics for Engineers. ....              | 2 $\frac{1}{4}$ × 5 $\frac{3}{8}$ inches, mor. | 1 00   |
| * Enlarged Edition, Including Tables. ....                                          | mor.                                           | 1 50   |
| Weisbach's Kinematics and the Power of Transmission. (Herrmann—Klein.) ....         | 8vo,                                           | 5 00   |
| Machinery of Transmission and Governors. (Herrmann—Klein.) ....                     | 8vo,                                           | 5 00   |
| Wood's Turbines. ....                                                               | 8vo.                                           | 2 50   |

## MATERIALS OF ENGINEERING.

|                                                                                                                     |               |      |
|---------------------------------------------------------------------------------------------------------------------|---------------|------|
| * Bovey's Strength of Materials and Theory of Structures. ....                                                      | 8vo,          | 7 50 |
| Burr's Elasticity and Resistance of the Materials of Engineering. ....                                              | 8vo,          | 7 50 |
| Church's Mechanics of Engineering. ....                                                                             | 8vo,          | 6 00 |
| * Greene's Structural Mechanics. ....                                                                               | 8vo,          | 2 50 |
| * Holley's Lead and Zinc Pigments. ....                                                                             | Large 12mo    | 3 00 |
| Holley and Ladd's Analysis of Mixed Paints, Color Pigments, and Varnishes. ....                                     | Large 12mo,   | 2 50 |
| Johnson's (C. M.) Rapid Methods for the Chemical Analysis of Special Steels, Steel-Making Alloys and Graphite. .... | Large 12mo,   | 3 00 |
| Johnson's (J. B.) Materials of Construction. ....                                                                   | 8vo,          | 6 00 |
| Keep's Cast Iron. ....                                                                                              | 8vo,          | 2 50 |
| Lanza's Applied Mechanics. ....                                                                                     | 8vo,          | 7 50 |
| Maire's Modern Pigments and their Vehicles. ....                                                                    | 12mo,         | 2 00 |
| Maurer's Technical Mechanics. ....                                                                                  | 8vo,          | 4 00 |
| Merriman's Mechanics of Materials. ....                                                                             | 8vo,          | 5 00 |
| * Strength of Materials. ....                                                                                       | 12mo,         | 1 00 |
| Metcalf's Steel. A Manual for Steel-users. ....                                                                     | 12mo,         | 2 00 |
| Sabin's Industrial and Artistic Technology of Paint and Varnish. ....                                               | 8vo,          | 3 00 |
| Smith's ((A. W.) Materials of Machines. ....                                                                        | 12mo,         | 1 00 |
| * Smith's (H. E.) Strength of Material. ....                                                                        | 12mo,         | 1 25 |
| Thurston's Materials of Engineering. ....                                                                           | 3 vols., 8vo, | 8 00 |
| Part I. Non-metallic Materials of Engineering. ....                                                                 | 8vo,          | 2 00 |
| Part II. Iron and Steel. ....                                                                                       | 8vo,          | 3 50 |
| Part III. A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents. ....                             | 8vo,          | 2 50 |
| Wood's (De V.) Elements of Analytical Mechanics. ....                                                               | 8vo,          | 3 00 |
| Treatise on the Resistance of Materials and an Appendix on the Preservation of Timber. ....                         | 8vo,          | 2 00 |
| Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel. ....                                | 8vo,          | 4 00 |

## STEAM-ENGINES AND BOILERS.

|                                                                    |       |      |
|--------------------------------------------------------------------|-------|------|
| Berry's Temperature-entropy Diagram. ....                          | 12mo, | 2 00 |
| Carnot's Reflections on the Motive Power of Heat. (Thurston.) .... | 12mo, | 1 50 |
| Chase's Art of Pattern Making. ....                                | 12mo, | 2 50 |

|                                                                                                         |              |       |
|---------------------------------------------------------------------------------------------------------|--------------|-------|
| Creighton's Steam-engine and other Heat Motors. ....                                                    | 8vo,         | 53 00 |
| Dawson's "Engineering" and Electric Traction Pocket-book. ...                                           | 16mo, mor.   | 5 00  |
| Ford's Boiler Making for Boiler Makers. ....                                                            | 18mo,        | 1 00  |
| * Gebhardt's Steam Power Plant Engineering. ....                                                        | 8vo,         | 6 00  |
| Goss's Locomotive Performance. ....                                                                     | 8vo,         | 5 00  |
| Hemenway's Indicator Practice and Steam-engine Economy. ....                                            | 12mo,        | 2 00  |
| Hutton's Heat and Heat-engines. ....                                                                    | 8vo,         | 5 00  |
| Mechanical Engineering of Power Plants. ....                                                            | 8vo,         | 5 00  |
| Kent's Steam boiler Economy. ....                                                                       | 8vo,         | 4 00  |
| Kneass's Practice and Theory of the Injector. ....                                                      | 8vo,         | 1 50  |
| MacCord's Slide-valves. ....                                                                            | 8vo,         | 2 00  |
| Meyer's Modern Locomotive Construction. ....                                                            | 4to,         | 10 00 |
| Moyer's Steam Turbine. ....                                                                             | 8vo,         | 4 00  |
| Peabody's Manual of the Steam-engine Indicator. ....                                                    | 12mo,        | 1 50  |
| Tables of the Properties of Steam and Other Vapors and Temperature-Entropy Table. ....                  | 8vo,         | 1 00  |
| Thermodynamics of the Steam-engine and Other Heat-engines. ...                                          | 8vo,         | 5 00  |
| Valve-gears for Steam-engines. ....                                                                     | 8vo,         | 2 50  |
| Peabody and Miller's Steam-boilers. ....                                                                | 8vo,         | 4 00  |
| Pupin's Thermodynamics of Reversible Cycles in Gases and Saturated Vapors. (Osterberg.) ....            | 12mo,        | 1 25  |
| Reagan's Locomotives: Simple, Compound, and Electric. New Edition. Large 12mo,                          |              | 3 50  |
| Sinclair's Locomotive Engine Running and Management. ....                                               | 12mo,        | 2 00  |
| Smart's Handbook of Engineering Laboratory Practice. ....                                               | 12mo,        | 2 50  |
| Snow's Steam-boiler Practice. ....                                                                      | 8vo,         | 3 00  |
| Spangler's Notes on Thermodynamics. ....                                                                | 12mo,        | 1 00  |
| Valve-gears. ....                                                                                       | 8vo,         | 2 50  |
| Spangler, Greene, and Marshall's Elements of Steam-engineering. ...                                     | 8vo,         | 3 00  |
| Thomas's Steam-turbines. ....                                                                           | 8vo,         | 4 00  |
| Thurston's Handbook of Engine and Boiler Trials, and the Use of the Indicator and the Prony Brake. .... | 8vo,         | 5 00  |
| Handy Tables. ....                                                                                      | 8vo,         | 1 50  |
| Manual of Steam-boilers, their Designs, Construction, and Operation                                     | 8vo,         | 5 00  |
| Manual of the Steam-engine. ....                                                                        | 2vols., 8vo, | 10 00 |
| Part I. History, Structure, and Theory. ....                                                            | 8vo,         | 6 00  |
| Part II. Design, Construction, and Operation. ....                                                      | 8vo,         | 6 00  |
| Wehrenfenning's Analysis and Softening of Boiler Feed-water. (Patterson.)                               | 8vo,         | 4 00  |
| Weisbach's Heat, Steam, and Steam-engines. (Du Bois.) ....                                              | 8vo,         | 5 00  |
| Whitham's Steam-engine Design. ....                                                                     | 8vo,         | 5 00  |
| Wood's Thermodynamics, Heat Motors, and Refrigerating Machines. .                                       | 8vo,         | 4 00  |

## MECHANICS PURE AND APPLIED.

|                                                                                     |            |       |
|-------------------------------------------------------------------------------------|------------|-------|
| Church's Mechanics of Engineering. ....                                             | 8vo,       | 6 00  |
| Notes and Examples in Mechanics. ....                                               | 12mo,      | 2 00  |
| Dana's Text-book of Elementary Mechanics for Colleges and Schools                   | 12mo,      | 1 50  |
| Du Bois's Elementary Principles of Mechanics:                                       |            |       |
| Vol. I. Kinematics. ....                                                            | 8vo,       | 3 50  |
| Vol. II. Statics. ....                                                              | 8vo,       | 4 00  |
| Mechanics of Engineering. Vol. I. ....                                              | Small 4to, | 7 50  |
| Vol. II. ....                                                                       | Small 4to, | 10 00 |
| * Greene's Structural Mechanics. ....                                               | 8vo,       | 2 50  |
| James's Kinematics of a Point and the Rational Mechanics of a Particle. Large 12mo, |            | 2 00  |
| * Johnson's (W. W.) Theoretical Mechanics. ....                                     | 12mo,      | 3 00  |
| Lanza's Applied Mechanics. ....                                                     | 8vo,       | 7 50  |
| * Martin's Text Book on Mechanics, Vol. I, Statics. ....                            | 12mo,      | 1 25  |
| * Vol. II, Kinematics and Kinetics. ....                                            | 12mo,      | 1 50  |
| Maurer's Technical Mechanics. ....                                                  | 8vo,       | 4 00  |
| * Merriman's Elements of Mechanics. ....                                            | 12mo,      | 1 00  |
| Mechanics of Materials. ....                                                        | 8vo,       | 5 00  |

|                                                  |             |        |
|--------------------------------------------------|-------------|--------|
| * Michie's Elements of Analytical Mechanics..... | 8vo,        | \$4 00 |
| Robinson's Principles of Mechanism.....          | 8vo,        | 3 00   |
| Sanborn's Mechanics Problems.....                | Large 12mo, | 1 50   |
| Schwamb and Merrill's Elements of Mechanism..... | 8vo,        | 3 00   |
| Wood's Elements of Analytical Mechanics.....     | 8vo,        | 3 00   |
| Principles of Elementary Mechanics.....          | 12mo,       | 1 25   |

## MEDICAL.

|                                                                                                                                                         |             |      |
|---------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|------|
| * Abderhalden's Physiological Chemistry in Thirty Lectures. (Hall and Defren.).....                                                                     | 8vo,        | 5 00 |
| von Behring's Suppression of Tuberculosis. (Bolduan.).....                                                                                              | 12mo,       | 1 00 |
| Bolduan's Immune Sera.....                                                                                                                              | 12mo,       | 1 50 |
| Bordet's Studies in Immunity. (Gay.).....                                                                                                               | 8vo,        | 6 00 |
| Davenport's Statistical Methods with Special Reference to Biological Variations.....                                                                    | 16mo, mor.  | 1 50 |
| Ehrlich's Collected Studies on Immunity. (Bolduan.).....                                                                                                | 8vo,        | 6 00 |
| * Fischer's Physiology of Alimentation.....                                                                                                             | Large 12mo, | 2 00 |
| de Fursac's Manual of Psychiatry. (Rosanoff and Collins.)....                                                                                           | Large 12mo, | 2 50 |
| Hammarsten's Text-book on Physiological Chemistry. (Mandel.)....                                                                                        | 8vo,        | 4 00 |
| Jackson's Directions for Laboratory Work in Physiological Chemistry.....                                                                                | 8vo,        | 1 25 |
| Lassar-Cohn's Practical Urinary Analysis. (Lorenz.).....                                                                                                | 12mo,       | 1 00 |
| Mandel's Hand-book for the Bio-Chemical Laboratory.....                                                                                                 | 12mo,       | 1 50 |
| * Pauli's Physical Chemistry in the Service of Medicine. (Fischer.)....                                                                                 | 12mo,       | 1 25 |
| * Pozzi-Escot's Toxins and Venoms and their Antibodies. (Cohn.)....                                                                                     | 12mo,       | 1 00 |
| Rostoski's Serum Diagnosis. (Bolduan.).....                                                                                                             | 12mo,       | 1 00 |
| Ruddiman's Incompatibilities in Prescriptions.....                                                                                                      | 8vo,        | 2 00 |
| Whys in Pharmacy.....                                                                                                                                   | 12mo,       | 1 00 |
| Salkowski's Physiological and Pathological Chemistry. (Orndorff.)....                                                                                   | 8vo,        | 2 50 |
| * Satterlee's Outlines of Human Embryology.....                                                                                                         | 12mo,       | 1 25 |
| Smith's Lecture Notes on Chemistry for Dental Students.....                                                                                             | 8vo,        | 2 50 |
| * Whipple's Typhoid Fever.....                                                                                                                          | Large 12mo, | 3 00 |
| * Woodhull's Military Hygiene for Officers of the Line.....                                                                                             | Large 12mo, | 1 50 |
| * Personal Hygiene.....                                                                                                                                 | 12mo,       | 1 00 |
| Worcester and Atkinson's Small Hospitals Establishment and Maintenance, and Suggestions for Hospital Architecture, with Plans for a Small Hospital..... | 12mo,       | 1 25 |

## METALLURGY.

|                                                                                                            |             |      |
|------------------------------------------------------------------------------------------------------------|-------------|------|
| Betts's Lead Refining by Electrolysis.....                                                                 | 8vo,        | 4 00 |
| Bolland's Encyclopedia of Founding and Dictionary of Foundry Terms used in the Practice of Moulding.....   | 12mo,       | 3 00 |
| Iron Founder.....                                                                                          | 12mo,       | 2 50 |
| " " Supplement.....                                                                                        | 12mo,       | 2 50 |
| Douglas's Untechnical Addresses on Technical Subjects.....                                                 | 12mo,       | 1 00 |
| Goesel's Minerals and Metals: A Reference Book.....                                                        | 16mo. mor.  | 3 00 |
| * Iles's Lead-smelting.....                                                                                | 12mo,       | 2 50 |
| Johnson's Rapid Methods for the Chemical Analysis of Special Steels, Steel-making Alloys and Graphite..... | Large 12mo, | 3 00 |
| Keep's Cast Iron.....                                                                                      | 8vo,        | 2 50 |
| Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.).....                                    | 12mo,       | 3 00 |
| Metcalf's Steel. A Manual for Steel-users.....                                                             | 12mo,       | 2 00 |
| Minet's Production of Aluminum and its Industrial Use. (Waldo.)....                                        | 12mo,       | 2 50 |
| * Ruer's Elements of Metallurgy. (Mathewson.).....                                                         | 8vo,        | 3 00 |
| Smith's Materials of Machines.....                                                                         | 12mo,       | 1 00 |
| Tate and Stone's Foundry Practice.....                                                                     | 12mo,       | 2 00 |
| Thurston's Materials of Engineering. In Three Parts.....                                                   | 8vo,        | 8 00 |
| Part I. Non-metallic Materials of Engineering, see Civil Engineering, page 9.                              |             |      |
| Part II. Iron and Steel.....                                                                               | 8vo,        | 3 50 |
| Part III. A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents.....                     | 8vo,        | 2 50 |

|                                                 |       |        |
|-------------------------------------------------|-------|--------|
| Ulke's Modern Electrolytic Copper Refining..... | 8vo,  | \$3 00 |
| West's American Foundry Practice.....           | 12mo, | 2 50   |
| Moulders' Text Book.....                        | 12mo, | 2 50   |

## MINERALOGY.

|                                                                                              |                          |       |
|----------------------------------------------------------------------------------------------|--------------------------|-------|
| Baskerville's Chemical Elements. (In Preparation.)                                           |                          |       |
| * Browning's Introduction to the Rarer Elements.....                                         | 8vo,                     | 1 50  |
| Brush's Manual of Determinative Mineralogy. (Penfield.).....                                 | 8vo,                     | 4 00  |
| Butler's Pocket Hand-book of Minerals.....                                                   | 16mo, mor.               | 3 00  |
| Chester's Catalogue of Minerals.....                                                         | 8vo, paper,              | 1 00  |
|                                                                                              | Cloth,                   | 1 25  |
| * Crane's Gold and Silver.....                                                               | 8vo,                     | 5 00  |
| Dana's First Appendix to Dana's New "System of Mineralogy".....                              | Large 8vo,               | 1 00  |
| Dana's Second Appendix to Dana's New "System of Mineralogy".....                             | Large 8vo,               |       |
| Manual of Mineralogy and Petrography.....                                                    | 12mo,                    | 2 00  |
| Minerals and How to Study Them.....                                                          | 12mo,                    | 1 50  |
| System of Mineralogy.....                                                                    | Large 8vo, half leather, | 12 50 |
| Text-book of Mineralogy.....                                                                 | 8vo,                     | 4 00  |
| Douglas's Untechnical Addresses on Technical Subjects.....                                   | 12mo,                    | 1 00  |
| Eakle's Mineral Tables.....                                                                  | 8vo,                     | 1 25  |
| Eckel's Stone and Clay Products Used in Engineering. (In Preparation.)                       |                          |       |
| Goessel's Minerals and Metals: A Reference Book.....                                         | 16mo, mor.               | 3 00  |
| Groth's Introduction to Chemical Crystallography (Marshall).....                             | 12mo,                    | 1 25  |
| * Hayes's Handbook for Field Geologists.....                                                 | 16mo, mor.               | 1 50  |
| Iddings's Igneous Rocks.....                                                                 | 8vo,                     | 5 00  |
| Rock Minerals.....                                                                           | 8vo,                     | 5 00  |
| Johannsen's Determination of Rock-forming Minerals in Thin Sections.....                     | 8vo,                     |       |
|                                                                                              | With Thumb Index         | 5 00  |
| * Martin's Laboratory Guide to Qualitative Analysis with the Blow-pipe.....                  | 12mo,                    | 1 50  |
|                                                                                              |                          | 60    |
| Merrill's Non-metallic Minerals: Their Occurrence and Uses.....                              | 8vo,                     | 4 00  |
| Stones for Building and Decoration.....                                                      | 8vo,                     | 5 00  |
| * Penfield's Notes on Determinative Mineralogy and Record of Mineral Tests.....              | 8vo, paper,              | 50    |
| Tables of Minerals, Including the Use of Minerals and Statistics of Domestic Production..... | 8vo,                     | 1 00  |
| * Pirsson's Rocks and Rock Minerals.....                                                     | 12mo,                    | 2 50  |
| * Richards's Synopsis of Mineral Characters.....                                             | 12mo, mor.               | 1 25  |
| * Ries's Clays: Their Occurrence, Properties and Uses.....                                   | 8vo,                     | 5 00  |
| * Ries and Leighton's History of the Clay-working Industry of the United States.....         | 8vo,                     | 2 50  |
| * Tillman's Text-book of Important Minerals and Rocks.....                                   | 8vo,                     | 2 00  |
| Washington's Manual of the Chemical Analysis of Rocks.....                                   | 8vo,                     | 2 00  |

## MINING.

|                                                                         |             |      |
|-------------------------------------------------------------------------|-------------|------|
| * Beard's Mine Gases and Explosions.....                                | Large 12mo, | 3 00 |
| * Crane's Gold and Silver.....                                          | 8vo,        | 5 00 |
| * Index of Mining Engineering Literature.....                           | 8vo,        | 4 00 |
|                                                                         | * 8vo, mor. | 5 00 |
| Mining Methods. (In Press.)                                             |             |      |
| Douglas's Untechnical Addresses on Technical Subjects.....              | 12mo,       | 1 00 |
| Eissler's Modern High Explosives.....                                   | 8vo,        | 4 00 |
| Goessel's Minerals and Metals: A Reference Book.....                    | 16mo, mor.  | 3 00 |
| Ihlseng's Manual of Mining.....                                         | 8vo,        | 5 00 |
| * Iles's Lead Smelting.....                                             | 12mo,       | 2 50 |
| Peele's Compressed Air Plant for Mines.....                             | 8vo,        | 3 00 |
| Riemer's Shaft Sinking Under Difficult Conditions. (Corning and Peele.) | 8vo,        | 3 00 |
| * Weaver's Military Explosives.....                                     | 8vo,        | 3 00 |
| Wilson's Hydraulic and Placer Mining. 2d edition, rewritten.....        | 12mo,       | 2 50 |
| Treatise on Practical and Theoretical Mine Ventilation.....             | 12mo,       | 1 25 |

## SANITARY SCIENCE.

|                                                                                                                |             |        |
|----------------------------------------------------------------------------------------------------------------|-------------|--------|
| Association of State and National Food and Dairy Departments, Hartford Meeting, 1906. ....                     | 8vo,        | \$3 00 |
| Jamestown Meeting, 1907. ....                                                                                  | 8vo,        | 3 00   |
| * Bashore's Outlines of Practical Sanitation. ....                                                             | 12mo,       | 1 25   |
| Sanitation of a Country House. ....                                                                            | 12mo,       | 1 00   |
| Sanitation of Recreation Camps and Parks. ....                                                                 | 12mo,       | 1 00   |
| Folwell's Sewerage. (Designing, Construction, and Maintenance.)....                                            | 8vo,        | 3 00   |
| Water-supply Engineering. ....                                                                                 | 8vo,        | 4 00   |
| Fowler's Sewage Works Analyses. ....                                                                           | 12mo,       | 2 00   |
| Fuertes's Water filtration Works. ....                                                                         | 12mo,       | 2 50   |
| Water and Public Health. ....                                                                                  | 12mo,       | 1 50   |
| Gerhard's Guide to Sanitary Inspections. ....                                                                  | 12mo,       | 1 50   |
| * Modern Baths and Bath Houses. ....                                                                           | 8vo,        | 3 00   |
| Sanitation of Public Buildings. ....                                                                           | 12mo,       | 1 50   |
| * The Water Supply, Sewerage, and Plumbing of Modern City Buildings. ....                                      | 8vo,        | 4 00   |
| Hazen's Clean Water and How to Get It. ....                                                                    | Large 12mo, | 1 50   |
| Filtration of Public Water-supplies. ....                                                                      | 8vo,        | 3 00   |
| Kinnicut, Winslow and Pratt's Purification of Sewage. (In Preparation.)                                        |             |        |
| Leach's Inspection and Analysis of Food with Special Reference to State Control. ....                          | 8vo,        | 7 50   |
| Mason's Examination of Water. (Chemical and Bacteriological)....                                               | 12mo,       | 1 25   |
| Water-supply. (Considered principally from a Sanitary Standpoint). ....                                        | 8vo,        | 4 00   |
| * Merriman's Elements of Sanitary Engineering. ....                                                            | 8vo,        | 2 00   |
| Ogden's Sewer Construction. ....                                                                               | 8vo,        | 3 00   |
| Sewer Design. ....                                                                                             | 12mo,       | 2 00   |
| Parsons's Disposal of Municipal Refuse. ....                                                                   | 8vo,        | 2 00   |
| Prescott and Winslow's Elements of Water Bacteriology, with Special Reference to Sanitary Water Analysis. .... | 12mo,       | 1 50   |
| * Price's Handbook on Sanitation. ....                                                                         | 12mo,       | 1 50   |
| Richards's Cost of Cleanness. ....                                                                             | 12mo,       | 1 00   |
| Cost of Food. A Study in Dietaries. ....                                                                       | 12mo,       | 1 00   |
| Cost of Living as Modified by Sanitary Science. ....                                                           | 12mo,       | 1 00   |
| Cost of Shelter. ....                                                                                          | 12mo,       | 1 00   |
| * Richards and Williams's Dietary Computer. ....                                                               | 8vo,        | 1 50   |
| Richards and Woodman's Air, Water, and Food from a Sanitary Standpoint. ....                                   | 8vo,        | 2 00   |
| * Richey's Plumbers', Steam-fitters', and Tinnerns' Edition (Building Mechanics Ready Reference Series). ....  | 16mo, mor,  | 1 50   |
| Rideal's Disinfection and the Preservation of Food. ....                                                       | 8vo,        | 4 00   |
| Sewage and Bacterial Purification of Sewage. ....                                                              | 8vo,        | 4 00   |
| Soper's Air and Ventilation of Subways. ....                                                                   | 12mo,       | 2 50   |
| Turneaure and Russell's Public Water-supplies. ....                                                            | 8vo,        | 5 00   |
| Venable's Garbage Crematories in America. ....                                                                 | 8vo,        | 2 00   |
| Method and Devices for Bacterial Treatment of Sewage. ....                                                     | 8vo,        | 3 00   |
| Ward and Whipple's Freshwater Biology. (In Press.)                                                             |             |        |
| Whipple's Microscopy of Drinking-water. ....                                                                   | 8vo,        | 3 50   |
| * Typhoid Fever. ....                                                                                          | Large 12mo, | 3 00   |
| Value of Pure Water. ....                                                                                      | Large 12mo, | 1 00   |
| Winslow's Systematic Relationship of the Coccaceæ. ....                                                        | Large 12mo, | 2 50   |

## MISCELLANEOUS.

|                                                                                                                  |            |      |
|------------------------------------------------------------------------------------------------------------------|------------|------|
| Emmons's Geological Guide-book of the Rocky Mountain Excursion of the International Congress of Geologists. .... | Large 8vo, | 1 50 |
| Ferrel's Popular Treatise on the Winds. ....                                                                     | 8vo,       | 4 00 |
| Fitzgerald's Boston Machinist. ....                                                                              | 18mo,      | 1 00 |
| Gannett's Statistical Abstract of the World. ....                                                                | 24mo,      | 75   |
| Haines's American Railway Management. ....                                                                       | 12mo,      | 2 50 |
| Hanausek's The Microscopy of Technical Products. (Winton) ....                                                   | 8vo,       | 5 00 |

|                                                                                   |                    |
|-----------------------------------------------------------------------------------|--------------------|
| <b>Jacobs's Betterment Briefs. A Collection of Published Papers on Or-</b>        |                    |
| <b>ganized Industrial Efficiency. ....</b>                                        | <b>8vo, \$3 50</b> |
| <b>Metcalfe's Cost of Manufactures, and the Administration of Workshops. ....</b> | <b>8vo, 5 00</b>   |
| <b>Putnam's Nautical Charts. ....</b>                                             | <b>8vo, 2 00</b>   |
| <b>Ricketts's History of Rensselaer Polytechnic Institute 1824-1894. ....</b>     |                    |
|                                                                                   | Large 12mo, 3 00   |
| <b>Rotherham's Emphasised New Testament. ....</b>                                 | Large 8vo, 2 00    |
| <b>Rust's Ex-Meridian Altitude, Azimuth and Star-finding Tables. ....</b>         | 8vo, 5 00          |
| <b>Standage's Decoration of Wood, Glass, Metal, etc. ....</b>                     | 12mo, 2 00         |
| <b>Thome's Structural and Physiological Botany. (Bennett). ....</b>               | 16mo, 2 25         |
| <b>Westermaier's Compendium of General Botany. (Schneider). ....</b>              | 8vo, 2 00          |
| <b>Winslow's Elements of Applied Microscopy. ....</b>                             | 12mo, 1 50         |

## HEBREW AND CHALDEE TEXT-BOOKS.

|                                                                               |                                  |
|-------------------------------------------------------------------------------|----------------------------------|
| <b>Gesenius's Hebrew and Chaldee Lexicon to the Old Testament Scriptures.</b> |                                  |
| <b>(Tregelles). ....</b>                                                      | <b>Small 4to, half mor, 5 00</b> |
| <b>Green's Elementary Hebrew Grammar. ....</b>                                | <b>12mo, 1 25</b>                |





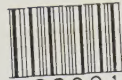


with  
pes

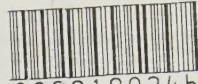
s



517.36 B16



a39001



006918034b

15379

